JEPPIAAR INSTITUTE OF TECHNOLOGY

"Self-Belief | Self Discipline | Self Respect"

DEPARTMENT

OF

COMPUTER SCIENCE AND ENGINEERING

LECTURE NOTES-MA8402

PROBABILITY AND QUEUING THEORY

(Regulation 2017)

Unit V

ADVANCED QUEUEING MODELS

(M/G/I):(GD) queue
Pollaczek-Khintchine formula
M/D/1 and M/EK/1 as special cases
Series queues
Open Jackson Networks

1 Non-Markovian queues and Queue Networking The M/G/1 queueing system (M/G/1) : (∞/GD) mdel Pollaczek Ishintchine Formula

Let N and N1 be the numbers of customers in the system at time t and t+T, when two consecutive customers have just left the system after getting service.

Let k be the no. of customer arriving in the system during the service time T.

$$N^1 = \begin{cases} K & \text{if } n = 0 \\ (N-1)+11 & \text{if } n > 0 \end{cases}$$

Where k = 0, 1, 2, 3, … is the no. of arrivals during the service time (K is a discrete random variable)

$$\text{Alternatively, if } \delta = \begin{cases} 1 & \text{if } N = 0 \\ 0 & \text{if } N < 0 \end{cases}$$

Then $N^1 = N-1+\delta+k$.

Various formula for (M/G/1) : (∞/GD)
Model can be summarized as follows:
1) Average no. of customer in the system

$$L_S = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}+\rho \qquad \text{where } \sigma^2 = V(T),\ P = \lambda E(T) \text{ (or) } \rho = \lambda/4$$

2) Average queue length

$$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$$

3) Average waiting time of a customer in the queue

$$W_q = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)}$$

4) Average waiting time of a customer spends in the system

$$W_S = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)}+\frac{1}{\mu}$$

Example :5.1.1

Automatic car wash facility operates with only one bay cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking let if the bay is busy. The parking lot is large enough to accommodate any no. of cars. If the service time for all cars is constant and equal to community determine.

(1) Mean no. of customers in the system LS
(2) Mean no. of customers in the queue Lq

(3) Mean waiting time of a customer in the system WS
(4) Mean waiting time of a customer in the system Wq

This is (M/G/I) : (∞/GD) model. Hence λ= 4 cars / hour.
T is the service time & is constant equal to 10 minutes

Then E(T) = 10 minutes & V(T) = 0.

$$\therefore \frac{1}{\mu} = 10 \Rightarrow \mu = \frac{1}{10} \text{ per minute}$$

$$\therefore \mu = 6 \text{ cars / hours} \quad \text{and } \sigma^2 = \text{Var (T)} = 0$$

$$\rho = \lambda/\mu = 4/6 = 2/3$$

Avg. no. of customers in the system

$$L_S = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho = 1.333 \square 1 \text{ car}.$$

Avg. No. of customers in the queue.

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = 0.667 \text{ cars.}$$

Avg. waiting time of a customer in the system

$$W_S = \frac{L_S}{\lambda} = 0.333 \text{ hour}$$

Avg. waiting time of a customer in the queue

$$W_q = \frac{L_q}{\lambda} = 0.167 \text{ hour}$$

Example :5.1.2 A car wash facility operates with only one day. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the factory's parking lot in the bay is busy. The parking lot is large enough to accommodate any no. of cars. If the service time for a car has uniform distribution b/w 8 & 12 minutes, find (i) The avg. no. of cars waiting in the parking lot and (ii) The avg. waiting time of car in the parking lot.

Solution

Solution

$$\lambda = 4; \quad \lambda = \frac{1}{15} \text{ cars / minutes}$$

E(T) = Mean of the uniform distribution in (8, 12)

$$= \frac{8+12}{2} = 10 \text{ minutes} \qquad \left[\text{Mean} = \frac{a+b}{2}\right]$$

$$V(T) = \frac{1}{2}(b-a)^2 = \frac{4}{3}$$

$$\Rightarrow \mu = \frac{1}{10} \text{cars / minutes and } \sigma^2 = 4/3$$

Then $\rho = \frac{\lambda}{\mu} = \frac{2}{3}$

By $\rho - k$ formula

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{\lambda(1-\rho)} = \frac{\frac{1}{225}\left(\frac{4}{3}\right) + \frac{4}{9}}{2\left(1-\frac{2}{3}\right)}$$

$$= \frac{\frac{4}{3}\left[\frac{1}{225} + \frac{1}{3}\right]}{\frac{2}{3}}$$

$= 0.675$ cars

The avg. no. of cars waiting in the parking lot = 0.675 crs The avg. waiting time of a car in the parking lot

$= L_q/\lambda = 0.675 \times 15$

$W_q = 10.125$ minutes .

2 QUEUE NETWORKS

Queue in series without possibility of queueing steady - state probability

P(0, 0) = Prob. (that both stages are empty

P(1, 0) = Prob. (that the 1st stage is full and the second is empty) P(1,1) = Prob. (that both the stages are full, first is working) P(b1, 1) = Prob. (that first is blocked and the second is full)

P(0, 1) = Prob. (that 1st stage is empty and the second is full)

2.1 STEADY STATE EQUATION

$$\lambda P(0,0) = \mu_2 P(0,1)$$
$$(\lambda + \mu_2) P(0,1) = \mu_1 P(0,1) + \mu_2 P(b,1)$$
$$\mu_1 P(1,0) = \mu_2 P(1,1) + \lambda P(0,0)$$
$$(\mu_1 + \mu_2) P(1,1) = \lambda P(0,1)$$
$$\mu_2 P(b,1) = \mu_1 P(1,1)$$
$$P(0,0) + P(0,1) + P(1,0) + P(1,1) + P(b,1) = 1$$

The solution of these equation is given by

$$P(0,0) = \lambda^2 \mu_1 (\mu_1 + \mu_2) / \sigma$$
$$P(0,1) = \lambda \mu_1 \mu_2 (\mu_1 + \mu_2) / \sigma$$
$$P(1,0) = \lambda \mu_2^2 (\lambda + \mu_1 + \mu_2) / \sigma$$
$$P(1,1) = \lambda^2 \mu_1 \mu_2 / \sigma$$
$$P(b,1) = \lambda^2 \mu_1^2 / \sigma$$

Where $\sigma = \mu_1 (\mu_1 + \mu_2)(\lambda^2 + \lambda\mu_2 + \mu_2^2) + \lambda(\mu_1 + \mu_2 + \lambda)\mu_2^2$

Hence the rate of loss - call is given by

$$L = P(1,0) + P(1,1) + P(0,1)$$

Example :5.2.1 A repair facility shared by a large no. of machines has two sequential stations with respective rates one per hour and two per hour. The cumulative failure rate of all machines is 0.5 per hour. Assuming that the system behaviour may be approximated by the two-stage tandem queue, determine (i) the avg repair time (ii) the prob. that both service stations are idle (iii) the station which is the bottleneck of the service facility.

Given

Given

$\lambda = 0.5 \qquad \mu_0 = 1 \qquad \mu_1 = 2$

$\rho_0 = \lambda/\mu_0 = 0.5 \qquad \& \qquad \rho_1 = \lambda/\mu_1 = 0.25$

The average length of the queue at station i(i = 0.1) is given by

$$E(N_i) = \frac{\rho_i}{1-\rho_i}$$

$\therefore\ E(N_0) = 1 \qquad \& \qquad E(N_1) = 1/3$

Using Little's formula, the repair delay at the two stations is respectively

$$E(R_0) = \frac{E(N_1)}{\lambda} = 2 \qquad \& \qquad E(R_1) = \frac{E(N_1)}{\lambda} = 2/3 \text{ hours}$$

Hence the avg. repair time is given by

$$E(R) = E(R_0) + E(R_1)$$
$$= 2 + 2/3 = 8/3 \text{ hours}$$

This can be decomposed into waiting time at station 0(=1 hour), the servi station $0\left(= 1/N_0 = 1\right)$, the waiting time at station $1\left(= 1/6 \text{ hour}\right)$ and the service time 1 $\left(1/\mu_1 = 1/2 \text{ hour}\right)$.

The prob. that both service stations are idle = P(0, 0) = $(1 - \rho_0)(1 - \rho_1)$

$= 3/8$

Station 0 is the bottleneck of the repair facility since $P_0 = 0.5$ is the largest value.

Open Central Service Queueing Model

$$E(N_j) = \frac{\rho_j}{1-\rho_j} \qquad \text{and} \qquad E(R_j) = \frac{1}{\lambda}\frac{\rho j}{1-\rho_j}$$

Example :5.2.2

Consider the open central server queueing model with two I/O channels with a common service rate of 1.2 sec^{-1}. The CPU service rate is 2 sec^{-1}, the arrival rate is 1/7 jobs / second. The branching prob. are given by P0 = 0.7, P1 = 0.3 and P2 = 0.6. Determine the steady state prob., assuming the service times are independent exponentially distributed random variables.

Determine the queue length distributions at each node as well as the avg. response time from the source on the sink.

We are given

$\mu 1$ = 1.2 / second

$\mu 2$ = 1.2 / second

We are given

μ_1 = 1.2 / second

μ_2 = 1.2 / second

μ_0 = 2 / second & $\lambda = 1/7$ jobs / second

The branching prob. are $P_0 = 0.7$, $P_1 = 0.3$ and $P_2 = 0.6$.

$$\lambda_0 = \frac{\lambda}{P_0}$$

$$\lambda_0 = \frac{1/7}{0.7}$$

$$= \frac{1}{4}9 = 10/49$$

$$\lambda_1 = \frac{\lambda P_1}{P_0} = 1/7 \text{ x } 0.3/0.7$$

$$= 3/49$$

$$\lambda_2 = \frac{\lambda P_2}{P_0} = 1/7 \text{ x } 0.6/0.7$$

$$= 6/49$$

The utilization ρ_j of node j is given by

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$$\rho_j = \frac{\lambda_i}{\mu_j} \quad (j = 0,1,2)$$

$$\rho_0 = \frac{\lambda_0}{\mu_0} = 5/49$$

$$\rho_1 = \lambda_1/\mu_1 = 5/98$$

$$\rho_2 = \lambda_2/\mu_2 = 5/49$$

The steady state prob. are given by

$$P_j(k_j) = (1-\rho_j)\rho_j^{k_j} \text{ at node j}$$

$$P_0(k_0) = \frac{44}{49}\left(\frac{5}{49}\right)k_0$$

$$P_1(k_1) = (1-\rho_1)\rho_1^{k_1}$$

$$= \frac{93}{98}\left(\frac{5}{98}\right)^{k_1}$$

$$P_2(k_2) = (1-\rho_2)\rho_2^{k_2}$$
$$= \frac{44}{49}\left(\frac{5}{49}\right)^{k_2}$$

The average queue length $E(N_j)$ of node j is given by

$$E(N_j) = \frac{\rho_j}{1-\rho_j}$$

$\therefore$ For node 0,

$$E(N_0) = \frac{\rho_0}{1-\rho_0} = \frac{5}{44} \text{ job / second.}$$

For node 1,

$$E(N_1) = \frac{\rho_1}{1-\rho_1} = \frac{5}{93} \text{ job / second.}$$

For node 2,

$$E(N_2) = \frac{\rho_2}{1-\rho_2} = \frac{5}{44} \text{ job / second.}$$

The average response time from the source to the sink is given by

$$E(R) = \frac{1}{\mu_0 P_0 - \lambda} + \sum_{j=1}^{m} \frac{1}{\frac{\mu_0 P_0}{\rho_j} - \lambda}$$

$$= \frac{1}{1.4 - \frac{1}{7}} + \frac{1}{\frac{7}{3}(1.2) - \frac{1}{7}} + \frac{1}{\frac{7}{6}(1.2) - \frac{1}{-7}}$$

$$= \frac{7}{8.8} + \frac{21}{55.8} + \frac{21}{26.4}$$

$= 0.7954 + 0.3763 + 0.7954$

$= 1.9671$

□ 1.97 seconds.

TUTORIAL PROBLEMS

1. Derive the Balance equation of the birth and death process.
2. Derive the Pollaczek-Khinchine formula.

3. Consider a single server, poisson input queue with mean arrival rate of 10hour currently the server works according to an exponential distribution with mean service time of 5minutes. Management has a training course which will result in an improvement in the variance of the service time but at a slight increase in the mean. After completion of the course;, its estimated that the mean service time will increase to 5.5 minutes but the standard deviation will decrease from 5 minutes to 4 minutes. Management would like to know; whether they should have the server undergo further training.

4. Ina heavy machine shop, the over head crane is 75%utilized.Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of 8.8 minutes.What is the average call ingrate for the services of the crane and what is the average delay in getting service? If the average service time is cut to 8.0 minutes, with standard deviation of 6.0 minutes, how much reduction will occur, on average, in the delay of getting served?

4. Automatic car wash facility operates with only on Bay. Cars arrive according toa Poisson process, with mean of 4 carsperhour and may wait in the facility's parking lot if the bay is busy. If the service time for all cars is constant and equal to 10 min, determine Ls, Lq,Ws and Wq

WORKED OUT EXAMPLES

Example :1 A car wash facility operates with only one day. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the factory's parking lot in the bay is busy. The parking lot is large enough to accommodate any no. of cars. If the service time for a car has uniform distribution b/w 8 & 12 minutes, find (i) The avg. no. of cars waiting in the parking lot and (ii) The avg. waiting time of car in the parking lot.

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= Lq/**λ =**0.675 x 15

Wq = 10.125 minutes .