JEPPIAAR INSTITUTE OF TECHNOLOGY

“Self-Belief | Self Discipline | Self Respect”

DEPARTMENT

OF

COMPUTER SCIENCE AND ENGINEERING

LECTURE NOTES-MA8402

PROBABILITY AND QUEUING THEORY

(Regulation 2017)

Unit III

RANDOM PROCESSES

- Introduction
- Classification
- stationary processes
- Markov processes
- Poisson processes
- Discrete parameter Markov chains
- Chapman Kolmogorov Equation
- Limiting distribution
- Random Telegraph processes

Introduction

In previews pages, we discussed about random variables. Random variable is a function of the possible outcomes of a experiment. But, it does not include the concept of time. In the real situations, we come across so many time varying functions which are random in nature. In electrical and electronics engineering, we studied about signals.

Generally, signals are classified into two types.

(i) Deterministic

(ii) Random

Here both deterministic and random signals are functions of time. Hence it is possible for us to determine the value of a signal at any given time. But this is not possible in the case of a random signal, since uncertainty of some element is always associated with it. The probability model used for characterizing a random signal is called a random process or stochastic process.

RANDOM PROCESS CONCEPT

A random process is a collection (ensemble) of real variable $\{X(s, t)\}$ that are functions of a real variable t where $s \in S$, S is the sample space and $t \in T$. (T is an index set).

REMARK

i) If t is fixed, then $\{X(s, t)\}$ is a random variable.

ii) If S and t are fixed $\{X(s, t)\}$ is a number.

iii) If S is fixed, $\{X(s, t)\}$ is a signal time function.

NOTATION

Here after we denote the random process $\{X(s, t)\}$ by $\{X(t)\}$ where the index set T is assumed to be continuous process is denoted by $\{X(n)\}$ or $\{X_n\}$.

A comparison between random variable and random process

Random Variable	Random Process
A function of the possible outcomes of an experiment is X(s)	A function of the possible outcomes of an experiment and also time i.e, X(s, t)
Outcome is mapped into a number x.	Outcomes are mapped into wave from which is a fun of time 't'.

Random Variable

A function of the possible outcomes of an experiment is X(s)

Outcome is mapped into a number x.

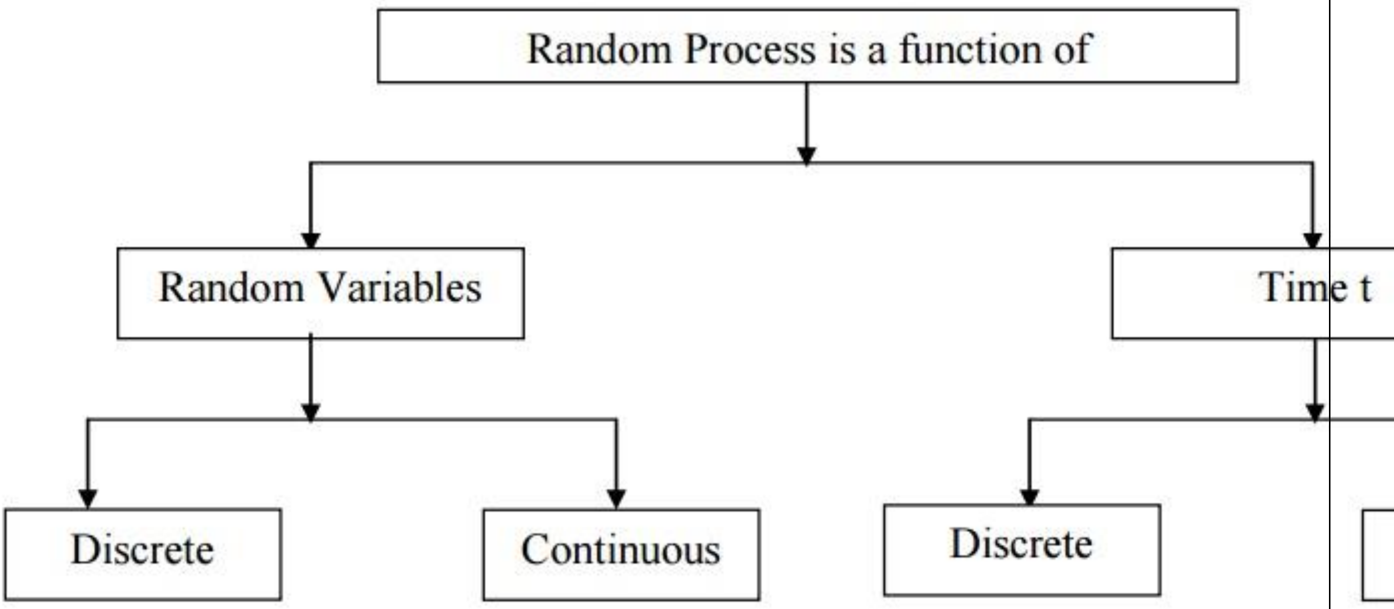
Random Process

A function of the possible outcomes of an experiment and also time i.e, X(s, t)

Outcomes are mapped into wave from which is a fun of time 't'.

CLASSIFICATION OF RANDOM PROCESSES

We can classify the random process according to the characteristics of time t and the random variable X = X(t) t & x have values in the ranges $-\infty < t < \infty$ and $-\infty < x < \infty$.



1 CONTINUOUS RANDOM PROCESS

If 'S' is continuous and t takes any value, then X(t) is a continuous random variable.

Example

Let X(t) = Maximum temperature of a particular place in (0, t). Here 'S' is a continuous set and $t \geq 0$ (takes all values), {X(t)} is a continuous random process.

2 DISCRETE RANDOM PROCESS

If 'S' assumes only discrete values and t is continuous then we call such random process {X(t) as Discrete Random Process.

Example

Let X(t) be the number of telephone calls received in the interval (0, t). Here, S = {1, 2, 3, …}

$T = \{t, t \geq 0\}$

$\therefore$ {X(t)} is a discrete random process.

3 CONTINUOUS RANDOM SEQUENCE

If 'S' is a continuous but time 't' takes only discrete is called discrete random sequence. **Example:** Let Xn denote the outcome of the n_{th} toss of a fair die.

Here S = {1, 2, 3, 4, 5, 6} T = {1, 2, 3, …}

∴ (Xn, n = 1, 2, 3, …} is a discrete random sequence.

CLASSIFICATION OF RANDOM PROCESSES BASED ON ITS SAMPLE

FUNCTIONS Non-Deterministic Process

A Process is called non-deterministic process if the future values of any sample function cannot be predicted exactly from observed values.

Deterministic Process

A process is called deterministic if future value of any sample function can be predicted from past values.

1 STATIONARY PROCESS

A random process is said to be stationary if its mean, variance, moments etc are constant. Other processes are called non stationary.

1. 1st Order Distribution Function of {X(t)}

For a specific t, X(t) is a random variable as it was observed earlier.

$F(x, t) = P\{X(t) \leq x\}$ is called the first order distribution of the process {X(t)}.

1st Order Density Function of {X(t)}

$$f(x,t) = \frac{\partial}{\partial x} F(x,t) \text{ is called the first order density of } \{X, t\}$$

2nd Order distribution function of {X(t)}

$F(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1;\ X(t_2) \le x_2\}$ is the point distribution of the random variables X(t1) and X(t2) and is called the second order distribution of the process {X(t)}.

2nd order density function of {X(T)}

$$f(x_1, x_2; t_1, t_2) = \frac{\partial^2 F(x_1, x_2; t_1, t_2)}{\partial x, \partial x_2} \text{ is called the second order density of } \{X(t)\}$$

2 First - Order Stationary Process

Definition

A random process is called stationary to order, one or first order stationary if its 1st order density function does not change with a shift in time origin.

In other words,

$f_X(x_1, t_1) = f_X(x_1, t_1 + C)$ must be true for any t1 and any real number C if {X(t1)} is to be a first order stationary process.

Example :3.3.1

Show that a first order stationary process has a constant mean.

Solution

Let us consider a random process {X(t1)} at two different times t1 and t2.

$$\therefore E\left[X(t_1)\right] = \int_{-\infty}^{\infty} x f(x, t_1) dx$$

[$f(x,t_1)$ is the density form of the random process $X(t_1)$]

$$\therefore E\left[X(t_2)\right] = \int_{-\infty}^{\infty} x f(x, t_2) dx$$

[$f(x,t_2)$ is the density form of the random process $X(t_2)$]

Let $t_2 = t_1 + C$

$$\therefore E\left[X(t_2)\right] = \int_{-\infty}^{\infty} x f(x, t_1 + C) dx = \int_{-\infty}^{\infty} x f(x, t_1) dx$$

$$= E\left[X(t_1)\right]$$

Thus $E\left[X(t_2)\right] = E\left[X(t_1)\right]$

Mean process {X(t1)} = mean of the random process {X(t2)}.

Definition 2:

If the process is first order stationary, then Mean = E(X(t)] = constant

4 Second Order Stationary Process

A random process is said to be second order stationary, if the second order density function stationary.

$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1 + C, t_2 + C)\ \forall x_1, x_2$ and C.

$E(X_1^2)$, $E(X_2^2)$, $E(X_1, X_2)$denote change with time, where

X = X(t1); X2 = X(t2).

5 Strongly Stationary Process

A random process is called a strongly stationary process or Strict Sense Stationary

Process (SSS Process) if all its finite dimensional distribution are invariance under translation of time 't'.

fX(x1, x2; t1, t2) = fX(x1, x2; t1+C, t2+C)

fX(x1, x2, x3; t1, t2, t3) = fX(x1, x2, x3; t1+C, t2+C, t3+C) In general fX(x1, x2..xn; t1, t2…tn) = fX(x1, x2..xn; t1+C, t2+C..tn+C) for any t1 and any real number C.

6 Jointly - Stationary in the Strict Sense

{X(t)} and Y{(t)} are said to be jointly stationary in the strict sense, if the joint distribution of X(t) and Y(t) are invariant under translation of time.

Definition Mean:

$$\mu_X(t) = E[X(t_1)], \quad -\infty < t < \infty$$

$\mu[X(t)]$ is also called mean function or ensemble average of the random process.

7Auto Correlation of a Random Process

Let X(t1) and X(t2) be the two given numbers of the random process {X(t)}. The auto correlation is

$$R_{XX}(t_1, t_2) = E\{X(t_1)\, x\, X(t_2)\}$$

Mean Square Value

Putting $t_1 = t_2 = t$ in (1), we get

$R_{XX}(t,t) = E[X(t)\, X(t)]$

$\Rightarrow \quad R_{XX}(t,t) = E[X^2(t)]$ is the mean square value of the random process.

8 Auto Covariance of A Random Process

$$C_{XX}(t_1, t_2) = E\{[X(t_1) - E(X(t_1))]\}[X(t_2) - E(X(t_2))]$$
$$= R_{XX}(t_1, t_2) - E[X(t_1)]E[X(t_2)]$$

Correlation Coefficient

The correlation coefficient of the random process {X(t)} is defined as

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\operatorname{Var} X(t_1)\, x \operatorname{Var} X(t_2)}$$

Where $C_{XX}(t_1, t_2)$ denotes the auto covariance.

Where CXX (t1, t2) denotes the auto covariance.

Statistical (Ensemble) Averages:

1. Mean = $E\left[X(t)\right] = \int_{-\infty}^{\infty} x f(x,t)\,dx$

2. Auto correlation fy of [x(t)]

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1,x_2,t_1,t_2)\,dx_1 dx_2$$

(or)

$$R_{XX}(t_1+t_2+\tau) = E\left[X(t)X(t+\tau)\right]$$

When τ = time difference = $t_2 - t_1$

3) Auto covariance of [X(t)]

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2).E\left[X(t_1)EX(t_2)\right]$$

$$C_{XX}(t,t) = E\left[X^2(t)\right] - E\left[X(t)\right]^2 \qquad \left[\because t_1 = t_2 = t\right]$$

= Var [X(t)]

4) Correlation coefficient of [X(t)]

$$\rho_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_2)C_{XX}(t_2,t_3)}}$$

Note : $\rho_{XX}(t,t) = 1$

5) Cross correlation

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$

(or)

$$R_{XY}(t,t+\tau) = E\left[X(t)Y(t+\tau)\right]$$

6) Cross Covariance

$$C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - E\left[X(t_1)\right]E\left[Y(t_2)\right]$$

Or $$C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - E\left[X(t)\right]E\left[Y(t+\tau)\right]$$

Cross Correlation Coefficient

$$\rho_{XY}(t_1,t_2) = \frac{C_{XY}(t_1,t_2)}{\sqrt{C_{XY}(t_1,t_1)C_{XY}(t_2,t_2)}}$$

FIRST ORDER STRICTLY STATIONARY PROCESS

Stationary Process (or) Strictly Stationary Process (or) Strict Sense Stationary Process [SSS Process]

A random process X(t) is said to be stationary in the strict sense, it its statistical characteristics do not change with time.
Stationary Process:

Formula: $E\left[X(t)\right] = \text{Cons tan t}$

$\gamma\left[X(t)\right] = \text{Cons tan t}$

1) Consider the RP X(t) = Cos (w0t +θ) where θis uniformly distributed in the interval -π to π. Check whether X(t) is stationary or not? Find the first and Second moments of the process.

Given X(t) = cos (W0t + θ)

Where θis uniformly distributed in (-π, π)

$$f(\theta) = \frac{1}{\pi-(-\pi)} - \frac{1}{2\pi}, \qquad -\pi < 0 < \pi$$

[from the def. of uniform distribution]

To prove (i) X(t) is SSS process
(ii) E[X(t)] = Constant
(iii) Var [X(t)] = Constant

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$$

$$= \int_{-\pi}^{\pi} \cos(w_0 t + \theta) . \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi}\left[\sin(w_0 t + \theta)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[\sin w_0 t + \pi] + \sin[\pi - w_0 t]$$

$$= \frac{1}{2\pi}[-\sin w_0 t + \sin w_0 t] = 0$$

$$E[X^2(t)] = E[ws^2(w_0 t + \theta)]$$

$$= \frac{1}{2} E[1 + \cos(2w_0 t + 2\theta)]$$

$$E[1] = \int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta = 1$$

$$E[\cos(2w_0 t + 2\theta)] = \int_{-\pi}^{\pi} \cos(2w_0 t + 2\theta) . \frac{1}{2}\pi$$

$$= \frac{1}{2\pi}\left[\sin \frac{(2\omega_0 t + 2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}[0] = 0$$

$$\therefore \Rightarrow E[X^2(t)] = \frac{1}{2}(1) + 0 = Y_2$$

$$\therefore \Rightarrow E[X^2(t)] = \frac{1}{2}(1) + 0 = Y_2$$

$$\text{Var}[X(t)] = E[X^2(t)] - [E[X(t)]]^2$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2} = \text{const}$$

$\therefore X(t)$ is a SSS Process./

S.T the RP X(t): Acos (w_0t +θ is not stationary if A and w0 are constants and θ is uniformly distributed random variable in (0, π). In X(t) = Acos (w_0t +θ

In ' θ 'uniformly distributed

$$f(\theta) = \frac{1}{\pi - 0} = \frac{1}{\pi} \qquad 0 < \theta < \pi$$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$$

$$= \int_0^{\pi} A\cos(w_0 t + \theta) \, {}^1\!/_{\pi} \, d\theta$$

$$= \frac{A}{\pi}\left[\sin(w_0 t + \theta)\right]_0^{\pi}$$

$$= -\frac{2A}{\pi} \sin w_0 t \neq \text{const.}$$

$\therefore$N(t) is not a stationary process.

$\therefore$N(t) is not a stationary process.

SECOND ORDER AND WIDE SENSE STATIONARY PROCESS

A Process is said to be second order stationary, if the second order density function statistics.

$$f(x_1, x_2 : t_1, t_2) = f(x_1, x_2 : t + \delta, t_2 + \delta), \ \forall x_1 \, x_2 \text{ and } \delta$$

If a random process X(t) is WSS then it must also be covariance stationary. In X(t) is WSS

i) E[X(t)] = μ= a const.

(ii) R(t1, t2) = a fy of (t 1 - t2) The auto covariance fn is gn by

The auto covariance fn is gn by

$$C(t_1, t_2) = R(t_1, t_2) - E[X(t_1) X(t_2)]$$

$$= R(t_1 - t_2) - E[X(t_1)] E[X(t_2)]$$

$$= R(t_1 - t_2) - \mu(\mu)$$

$$= R(t_1 - t_2) - \mu^2$$

Which depends only on the time difference. Hence X(t) is covariance stationary.
If X(t) is a wide sense stationary process with auto correlation R ()τ =Ae$^{-\alpha()}$, determine the second order moment of the random variable X(8) - X(5).

Given $R(\tau) = Ae^{-\alpha(\tau)}$

$R(t_1, t_2) = Ae^{-\alpha(t_1 - t_2)}$

$E\left[X^2(t)\right] = R(t,t) = A$

$E\left[X^2(8)\right] = A$

$E\left[X^2(5)\right] = A$

$E\left[X(8)X(15)\right] = R|8,5| = Ae^{-\alpha}(8,5)$

$= Ae^{-3\alpha}$

The second moment of X(8) - X(5) is given by

$E\left[X(8) - X(15)\right]^2 = E\left[X^2(8)\right]^2 + E\left[X^2(5)\right] - 2E\left[X(8)X(5)\right]$

The second moment of X(8) - X(5) is given by

$= A + A - 2Ae^{-3\alpha}$

$= 2A(1 - e^{-3\alpha})$

Example:3.3.1

S.T the random process $X(t) = A\cos(w_t+\theta)$ is wide sense stationary if A & w are constants and θ is uniformly distributed random variable in $(0, 2\pi)$.

Given $X(t) = A\cos(wt + \theta)$

$$f(\theta) = \frac{1}{2\pi - 0} = \begin{cases} \frac{1}{2\pi}, & 0 < \theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$f(\theta) = 1/2\pi$$

To prove X(t) is WSS

(i) $E[X(t)]$=Constant

(ii) $R(t_1, t_2) = \text{a fn of } (t_1 - t_2)$

(i) $E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$

$$\Rightarrow E[A\cos(wt+\theta)] = \int_{-\infty}^{\infty} A\cos(wt+\theta) f(\theta) d\theta$$

$$= \text{constant}$$

(ii) $R(t_1, t_2) = E[X(t_1)x(t_2)]$

$$= E[A\cos(wt_1\theta) A\cos(wt_2 + \theta)]$$

$$= E[A^2\cos(wt_1 + \theta)\cos(wt_2 + \theta)]$$

$$= \frac{A^2}{2} E[\cos w(t_1 + t_2) + 2\theta] + \frac{A^2}{2}\cos[w(t_1 - t_2)]$$

$$= \frac{A^2}{2} E\left[\cos w\left(t_1 + t_2\right) + 2\theta\right] + \frac{A^2}{2} \cos\left[w\left(t_1 - t_2\right)\right]$$

$$= 0$$

$$\Rightarrow R\left(t_1, t_2\right) = \frac{A^2}{2} \cos\left[w\left(t_1 - t_2\right)\right]$$

$$= \text{a fn of } (t_1 - t_2)$$

Hence X(t) is a WSS Process.

Example3.3.2

If $X(t) = A \cos \lambda t + B \sin \lambda t$, where A and B are two independent normal random variable with $E(A) = E(B) = 0$, $E(A^2) = E(B^2) = \sigma^2$, where λ is a constant. Prove that $\{X(t)\}$ is a Strict Sense Stationary Process of order 2 (or)

If $X(t) = A \cos \lambda t + B \sin \lambda t$, $t \geq 0$ is a random process where A and B are independent $N(0, \sigma^2)$ random variable, Examine the WSS process X(t).

Given $X(t) : A\cos\lambda t + B\sin\lambda t$ --------------- (1)

$E(A) = 0;$ $E(B) = 0$ --------------- (2)

$$E\left(A^2\right)=\sigma^2=k;\ E\left(B^2\right)=\sigma^2=k$$

$$E[AB]=E[A]E[B] \qquad [\because A \ \&\ B \text{ are independent}]$$

$$=0$$

To prove : X(t) is WSS Process

i.e. (i) $E\left[X(t)\right]=\text{Constant}$

(ii) $R\left(t_1,t_2\right)=\text{a fn of }(t_1 - t_2)$

$$E\left[X(t)\right] \quad = E\left[A\cos\lambda t + B\sin\lambda t\right]$$

$$= \cos\lambda t E\left[A\right]\sin\lambda t E\left(B\right)$$

$$=0$$

$$R\left(t_1,t_2\right)=E\left[X\left(t_1\right)X\left(t_2\right)\right]$$

$$= E\left[\left(A\cos\lambda t_1 + B\sin\lambda t_1\right)\left(A\cos\lambda t_2 + B\sin\lambda t_2\right)\right]$$

$$= E\left[A^2\cos\lambda t_1\cos\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2 + \right.$$

$$\left. AB\cos\lambda t_1\sin\lambda t_2 + \sin\lambda t_1\cos\lambda t_2\right]$$

$$=\cos\lambda t_1\cos\lambda t_2 E\left[A^2\right]+\sin\lambda t_1\sin\lambda t_2 E\left[B^2\right]$$

$$+E\left[AB\right]\left[\sin\left(\lambda t_1+\lambda t_2\right)\right]$$

$$=K\cos\lambda t_1\cos\lambda t_2 + K\sin\lambda t_1\sin\lambda t_2 + 0$$

$$= K\cos\lambda\left(t_1-t_2\right)$$

$$=\text{a fn of }(t_1 - t_2)$$

Both the conditions are satisfied. Hence X(t) is a WSS Process.

Example:3.3.3

Consider a random process X(t) defined by $N\left(t\right)=U\cos t+\left(V+1\right)\sin t$, where U and V are independent random variables for which E(U) = E(V) = 0, $E(U^2) = E(V^2) = 1$. Is X(t) is WSS? Explain your answer?

Given $X\left(t\right)=U\cos t+\left(v+1\right)\sin t$

$$E(U) = E(V) = 0$$
$$E(U^2) = E(V^2) = 1$$
$$E(UV) = E(U)\,E(V) = 0$$

$$E\left[X(t)\right] \quad = E\left[V\cos t+\left(V+1\right)\sin t\right]$$

$$= E\left(U\right)\cos t + E\left(V\right)\sin t + \sin t$$

$$= 0 + 0 + \sin t$$

$$= \sin t$$

$$\neq \text{a constant}$$

$\Rightarrow X\left(t\right)$ is not a WSS Process.

CROSS CORRELATION

The cross correlation of the two random process {X(t)} and {Y(t)} is defined by RXY (t1, t2) = E[X(t1) Y (t2)]

WIDE - SENSE STATIONARY (WSS)

A random process {X(t)} is called a weakly stationary process or covariance stationary process or wide-sense stationary process if

i) E{X(t)} = Constant

ii) E[X(t) X(t+τ] = Rxx(τ) depend only on τ when τ = t2 - t1.

REMARKS :

SSS Process of order two is a WSS Process and not conversely.

EVOLUTIONARY PROCESS

A random process that is not stationary in any sense is called as evolutionary process.

SOLVED PROBLEMS ON WIDE SENSE STATIONARY PROCESS

Example:3.6.1

Given an example of stationary random process and justify your claim.

Solution:

Let us consider a random process X(t) = A as (wt + θ) where A &ω are custom and 'θ' is uniformlydistribution random Variable in the interval (0, 2π).

Since 'θ' is uniformly distributed in (0, 2π), we have

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, 0 < C < 2\pi \\ 0 \text{ , otherwise} \end{cases}$$

$$\therefore E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$$

$$= \int_{0}^{2\pi} A\,\omega(\omega t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi} \left[\sin(\omega t + \theta)\right]_{0}^{2\pi}$$

$$= \frac{A}{2\pi} \left[\text{Sin}(2\pi + \omega t) - \text{Sin}(\omega t + 0)\right]$$

$$= \frac{A}{2\pi} [\text{Sin}\omega t - \sin \omega t]$$

$$= 0 \text{ constant}$$

Since E[X(t)] = a constant, the process X(t) is a stationary random process.

Example:3.6.2 which are not stationary

Examine whether the Poisson process {X(t)} given by the probability law P{X(t)=n] =

$$\frac{e^{-\lambda t}(\lambda t)}{\lfloor n}, \; n = 0, 1, 2, \ldots$$

Solution

We know that the mean is given by

$$E\left[X(t)\right] = \sum_{n=0}^{\infty} nP_n(t)$$

$$= \sum_{n=0}^{\infty} \frac{ne^{-\lambda t}(\lambda t)^n}{\lfloor n}$$

$$= \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{\lfloor n-1}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{\lfloor n-1}$$

$$= e^{-\lambda t}\left[\frac{\lambda t}{0!} + \frac{(\lambda t)^2}{1!} + \ldots\right]$$

$$= (\lambda t)e^{-\lambda t}\left(1 + \frac{\lambda t}{\lfloor 1} + \frac{(\lambda t)^2}{\lfloor 2} + \ldots\right)$$

$$= (\lambda t)e^{-\lambda t}e^{\lambda t}$$

$$= \lambda t \text{, depends on t}$$

Hence Poisson process is not a stationary process.

Hence Poisson process is not a stationary process.

ERGODIC PROCESS

Ergodic Process are processes for which time and ensemble (statistical) averages are interchangeable the concept of ergodicity deals with the equality of time and statistical average.

Time Average

It X(t) is a random process, then $\frac{1}{2\pi}\int_{-T}^{T} X(t)dt$ is called the time ave

X(t) over (-T, T) and is denoted by $\overline{X}_T$.

Note

1. $\overline{X}_T = \frac{1}{2\pi}\int_{-T}^{T} X(t)dt$, X(t) defined in (−T, T)

2. $\overline{X}_T = \frac{1}{T}\int_{0}^{T} X(t)dt$, D(t) defined in (0, T)

MARKOV PROCESS - MARKOV CHAINS

1 Markov Process

A random process X(t) is said to be Markov Process, it

$$P\left[X(t) \le x/_{X(t_1)} - x_1, X(t_2) = x_2 \ldots X(t_n) = x_n\right]$$

$$= P\left[X(t) \le x/_{X}(t_n) = x_n\right]$$

2 Markov Chain

A discrete parameter Markov Process is called Markov Chain.

If the tmp of a Markov Chain is $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$, find the steady state distribution of the chain.

Given $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

If $\pi = (\pi_1 \pi_2)$ is the steady state distribution of the chain, then by the property of π, we have

$$\pi P = \pi \qquad ----------(1)$$

$$\pi_1 + \pi_2 = 1 \qquad ----------(2)$$

$$\Rightarrow (\pi_1\ \pi_2)\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = (\pi_1 \pi_2)$$

$$\left[\pi_1[0] + \pi_2(1/2) + \pi_1(1) + \pi_2(1/2)\right] = [\pi_1\ \pi_2]$$

$$\tfrac{1}{2}\pi_2 = \pi_1 \qquad ----------- (3)$$

$$\pi_1 + \tfrac{1}{2}\pi_2 = \pi_2 \qquad --------- (4)$$

$$\pi_1 + \pi_2 = 1$$

$$\tfrac{1}{2}\pi_2 + \pi_2 = 1 \Rightarrow \tfrac{3}{2}\pi_2 = 1 \quad \text{by (3)}$$

$$\pi_2 = \tfrac{2}{3}$$

$$(3) \Rightarrow \pi_1 = \tfrac{1}{2}\pi_2 = \tfrac{1}{2}\left(\tfrac{2}{3}\right) = \tfrac{1}{3}$$

$\therefore$ The steady state distribution of the chain is $\pi = (\pi_1\ \pi_2)$

i.e. $\pi = \left(\tfrac{1}{3}\ \tfrac{2}{3}\right)$

Example :3.4.1

An Engineering analysing a series of digital signals generated by a testing system observes that only 1 out of 15 highly distorted signals followed a highly distorted signal with no recognizable signal, where as 20 out of 23 recognizable signals follow recognizable signals with no highly distorted signals b/w. Given that only highly distorted signals are not recognizable. Find the fraction of signals that are highly distorted.

π_1= The fraction of signals that are recognizable [R]
π_2 = The fraction of signals that are highly distorted [D]

The tmp of the Markov Chain is

$$P = \begin{array}{c} \\ R \\ D \end{array}\begin{array}{c} \begin{array}{cc} R & D \end{array} \\ \begin{bmatrix} \tfrac{20}{23} & - \\ - & \tfrac{1}{15} \end{bmatrix} \end{array} \Rightarrow P = \begin{bmatrix} \tfrac{20}{23} & \tfrac{3}{23} \\ \tfrac{14}{15} & \tfrac{1}{15} \end{bmatrix}$$

- 1 out of 15 highly distorted signals followed a highly distorted signal with no recognizable signal

$$[P(D \to D)] = \tfrac{1}{15}$$

- 20 out of 23 recognizable signals follow recognizable signals with no highly distorted signals.
- If the tmp of a chain is a stochastic martin, then the sum of all elements of any row is equal to 1.

If $\pi = (\pi_1 \pi_2)$ is the steady state distribution of the chain, then by the property of π, we have

$$\pi P = \pi$$

$$\pi_1 + \pi_2 = 1$$

$$\Rightarrow (\pi_1\, \pi_2)\begin{bmatrix} 20/23 & 3/23 \\ 14/15 & 1/15 \end{bmatrix} = (\pi_1 \pi_2)$$

$$20/23\, \pi_1 + 14/15\, \pi_2 = \pi_1 \qquad -----(3)$$

$$3/23\, \pi_1 + 1/15\, \pi_2 = \pi_2 \qquad -----(4)$$

(3) $\pi_2 = 45/322\, \pi_1$

$\Rightarrow \pi_1 = 322/367$

(2)

$\pi_2 = 45/367$

$\therefore$ The steady state distribution of the chain is

$\pi = (\pi_1 \pi_2)$

i.e. $\pi = \left(\dfrac{322}{367} \quad \dfrac{45}{367} \right)$

$\therefore$ The fraction of signals that are highly distorted is $45/367$

$\Rightarrow \dfrac{45}{367} \text{x} 100\% = 12.26\%$

Example :3.4.2

Transition prob and limiting distribution. A housewife buys the same cereal in successive weeks. If the buys cereal A, the next week she buys cereal B. However if she buys B or C, the next week she is 3 times as likely to buy A as the other cereal. In the long run how often she buys each of the 3 cereals.

Given : Let $\pi_1 \rightarrow$ Cereal A

$\pi_2 \rightarrow$ Cereal B

$\pi_3 \rightarrow$ Cereal C

$\therefore$ the tpm of the Markov Chain is

$$P = \begin{bmatrix} 0 & 1 & - \\ 3/4 & 0 & - \\ 3/4 & - & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix}$$

If $\pi = (\pi_1, \pi_2, \pi_3)$ is the steady - state distribution of the chain then by the property of π we have,

$$\pi P = \pi$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\Rightarrow (\pi_1\ \pi_2\ \pi_3)\begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} = [\pi_1\ \pi_2\ \pi_3]$$

$$\tfrac{3}{4}\pi_2 + \tfrac{3}{4}\pi_3 = \pi_1 \qquad (3)$$

$$\pi_1 + \tfrac{1}{4}\pi_3 = \pi_2 \qquad (4)$$

$$\tfrac{1}{4}\pi_2 = \pi_3 \qquad (5)$$

$$(3) \Rightarrow \tfrac{3}{4}\pi_2 + \tfrac{3}{4}\left(\tfrac{1}{4}\pi_2\right) = \pi_1 \qquad \text{(by 5)}$$

$$\tfrac{15}{16}\pi_2 = \pi_2 \qquad (6)$$

$$(2) \Rightarrow \pi_1 + \pi_2 + \pi_3 = 1$$

$$\tfrac{15}{16}\pi_2 + \pi_2 + \frac{1}{4}\pi_2 = 1 \qquad \text{by (5) \& (6)}$$

$$\frac{35}{16}\pi_2 = 1$$

$$\pi_2 = \frac{16}{35}$$

$$(6) \Rightarrow \pi_1 = \frac{15}{16}\pi_2$$

$$\pi_1 = \frac{15}{35}$$

$$(5) \Rightarrow \pi_3 = \frac{1}{4}\pi_2$$

$$\pi_3 = \frac{4}{35}$$

$\therefore$ The steady state distribution of the chain is

$$\pi = (\pi_1\ \pi_2\ \pi_3)$$

$$\text{i.e.,}\ \pi = \left(\tfrac{15}{35}\ \ \tfrac{16}{35}\ \ \tfrac{4}{35}\right)$$

n - step tmp P^n

Example : 3.4.1 The transition Prob. Martix of the Markov Chain $\{X_n\}$, n = 1,2,3,… having 3 states 1, 2 & 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $P^{(0)} = (0.7\ \ 0.2\ \ 0.1)$.

Find (i) $P(X_2 = 3)$ and (ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Solution

Given $P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$.

$$\Rightarrow \quad P[X_0 = 1] = 0.7$$

$$P(X_0 = 2) = 0.2$$

$$P[X_0 = 3] = 0.1$$

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}^{(1)} & P_{12}^{(1)} & P_{13}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & P_{23}^{(1)} \\ P_{31}^{(1)} & P_{32}^{(1)} & P_{33}^{(1)} \end{bmatrix}$$

$$P^2 = P \, . \, P$$

$$= \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$$

$$\text{(i) } P[X_2 = 3] = \sum_{i=1}^{3} P[X_2 = 3 / X_0 = i] P[\lambda_0 = i]$$

$$= P[X_2 = 3 / X_0 = 1] P[X_0 = 1] + P[X_2 = 3 / X_0 = 3] P[X_0 = 2] +$$

$$P[X_2 = 3 / X_{0=3}] P[X_0 = 3]$$

$$= P_{13}{}^{(2)} P[X_0 = 1] + P_{23}{}^{(2)} P[X_0 = 2] + P_{33}{}^{(2)} P[X_0 = 3]$$

$$= (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1)$$

$$= 0.279$$

$$\text{(ii) } P[X_3 = 2,\ X_2 = 3,\ X_1 = 3,\ X_0 = 2]$$

$$= P_{32}{}^{(1)} P[X_2 = 3 / X_1 = 3, X_0 = 2] P[\lambda_1 = 3, X_0 = 2]$$

$$= P_{32}{}^{(1)}\ P_{33}{}^{(1)}\ P_{23}{}^{(1)}\ P[X_0 = 2]$$

$$= (0.4)\ (0.3)\ (0.2)\ (0.2)$$

$$= 0.0048$$

TYPE 5

A training process is considered as two State Markov Chain. If it rain, it is considered to be state 0 & if it does not rain the chain is in stable 1. The tmp of the Markov Chain is defined as

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

i. Find the Prob. That it will rain for 3 days from today assuming that it is raining today.

ii. Find also the unconditional prob. That it will rain after 3 days with the initial Prob. Of state) and state 1 as 0.4 & 0.6 respectively.

Solution:

Given $P=\begin{bmatrix}0.6 & 0.4\\0.2 & 0.8\end{bmatrix}$

$$P^{(2)} = P^2 = \begin{bmatrix}0.6 & 0.4\\0.2 & 0.8\end{bmatrix}\begin{bmatrix}0.6 & 0.4\\0.2 & 0.8\end{bmatrix}$$

$$= \begin{bmatrix}0.44 & 0.56\\0.28 & 0.72\end{bmatrix}$$

$$P^{(3)} = P^3 = P^2P$$

$$= \begin{bmatrix}0.376 & 0.624\\0.312 & 0.688\end{bmatrix}$$

(i) If it rains today, then Prob. Distribution for today is (1 0)

$$\therefore P(\text{after 2 days}) = \begin{pmatrix}1 & 0\end{pmatrix}\begin{bmatrix}0.376 & 0.624\\0.312 & 0.688\end{bmatrix}$$

$$= [0.376 \quad 0.624]$$

$\therefore$ P [Rain for after 3 days] $= 0.376$

(ii) Given $P^{(0)} = \begin{pmatrix}0.4 & 0.6\end{pmatrix}$

$$P[\text{after 3 days}] = \begin{pmatrix}0.4 & 0.6\end{pmatrix}\begin{bmatrix}0.376 & 0.624\\0.312 & 0.688\end{bmatrix}$$

$$= \begin{pmatrix}0.3376 & 0.6624\end{pmatrix}$$

$\therefore$ P[rain for after 3 days] $= 0.3376$

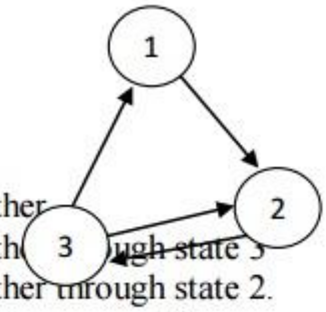
Example :3.5.1

Prove that the matrix $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\\frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}$ is the tpm of an irreducible Markov Chain?

(or)

Three boys A, B, C are throwing a ball each other. A always throws the ball to B & B always throws the ball to C but C is just as like to throw the ball to B as to A. State that the process is Markov Chain. Find the tpm and classify the status.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$



(a) let $X_n = \{1,2,3\} \Rightarrow$ finite

State 2 & 3 are communicate with each other
State 1 & 2 are communicate with each other through state 3
State 3 & 1 are communicate with each other through state 2.

$\Rightarrow$ The Markov Chain is irreducible (3)

From (1) & (2) all the states are persistent and non-null (3)

One can get back to

State 1 in	$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$	(3 steps)
State 2 in	$2 \rightarrow 3 \rightarrow 2$	(2 steps)
State 2 in	$2 \rightarrow 3 \rightarrow 1 \rightarrow 2$	(3 steps)
State 3 in	$3 \rightarrow 1 \rightarrow 2 \rightarrow 3$	(3 steps)
State 3 in	$3 \rightarrow 2 \rightarrow 2$	(2 steps)

$\Rightarrow$ The states are aperiodic (4)

[$\because$ The states are not periodc]

From (3) & (4) we get all the states are Ergodic.

ERGODIC RANDOM PROCESS

Time Average

The time average of a random process {X(t)} is defined as

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

Ensemble Average

The ensemble average of a random process {X(t)} is the expected value of the random variable X at time t

Ensemble Average = E[X(t)]

Ergodic Random Process

{X(t)} is said to be mean Ergodic

$$\text{If } \lim_{T\to\infty} \overline{X_T} = \mu$$

$$\Rightarrow \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,dt = \mu$$

Mean Ergodic Theorem

Let {X(t)} be a random process with constant mean μ and let XT be its time average. Then {X(t)} is mean ergodic if

$$\lim_{T\to\infty} \operatorname{Var} \overline{X_T} = 0$$

Correlation Ergodic Process

The stationary process {X(t)} is said to be correlation ergodic if the process {Y(t)} is mean ergodic where

Y(t) = X(t) X(t+λ)

$$E\left[y(t)\right] = \lim_{|T|\to\infty} \overline{Y_T} \text{ when } \overline{Y_T} \text{ is the time average of } Y(t).$$

MARKOV PROCESS

Definition

A random process {X(t)} is said to be markovian if

$$P\left[X(t_{n+1}) \le X_{n+1} / X(n) + x_n, x(t_{n-1}) = x_{n-1} \ldots x(t_0 = x_0)\right]$$
$$P\left[X(t_{n+1}) \le X_{n+1} / x(t_n) = x_n\right]$$

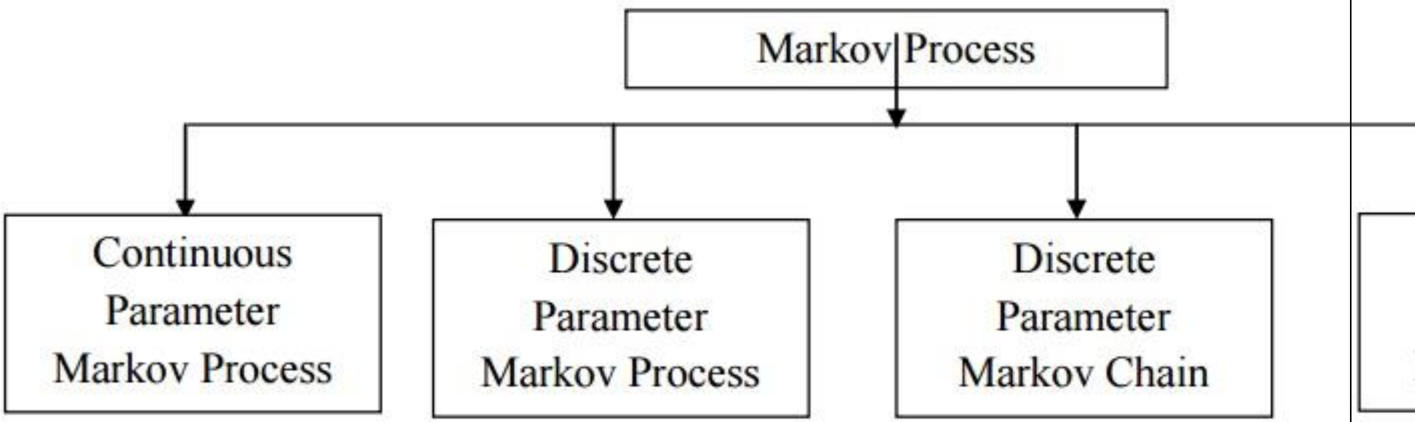
$$\text{Where } t_0 \le t_1 \le t_2 \le \ldots \le t_n \le t_{n+1}$$

Examples of Markov Process

1.The probability of raining today depends only on previous weather conditions existed for the last two days and not on past weather conditions.

2.A different equation is markovian.

Classification of Markov Process



MARKOV CHAIN

Definition

We define the Markov Chain as follows

If P{Xn = an/Xn-1 = an-1, Xn-2 = an-2, … X0 = a0}

⇒P{Xn = an / Xn-1 = an-1} for all n. the process {Xn}, n = 0, 1, 2… is called as Markov Chains.

1.a1, a2, a3, … an are called the states of the Markov Chain.

2.The conditional probability $P\{X_n = a_j \mid X_{n-1} = a_i\} = P_{ij}(n-1, n)$ is called the one step

transition probability from state a_i to state a_j at the n^{th} step. 3.The tmp of a Markov chain is a stochastic matricx

i) $P_{ij} \geq 0$

ii) ΣPij = 1 [Sum of elements of any row is 1]

Poisson Process

The Poisson Process is a continuous parameter discrete state process which is very useful model for many practical situations. It describe number of times occurred. When an experiment is conducted as a function of time.

Property Law for the Poisson Process

Let λ be the rate of occurrences or number of occurrences per unit time and Pn(t) be the probability of n occurrences of the event in the interval (0, t) is a Poisson distribution with parameter λt.

$$\text{i.e. } P\left[X(t)=n\right]=\frac{e^{-\lambda t}(\lambda t)^n}{n!},\quad n=0,1,2,\ldots$$

$$P_n(t)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Second Order Probability Function of a Homogeneous Poisson Process

$$P\left[X(t_1)\right]=n_1=\left[X(t_2)=n_2\right]=P\left[X(t_1=n_1)\right].P\left[X(t_2=n_2)\right]/$$

$$\left[X(t_1)=n_2\right],t_2>t_1$$

$$=P\left[X(t_1)=n_1\right].P\quad \text{[the even occurs } n_2\text{-n times in the interval } (t_2=t_1)$$

$$=\frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{\lfloor n_1}.\frac{e^{-\lambda(t_2-t_1)}\left\{\lambda(t_2-t_1)\right\}^{n_2=n_1}}{\lfloor n_2-n_1},n_2\geq n_1$$

$$=\begin{cases}\dfrac{e^{-\lambda t_2}.\lambda^{n_2}.t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{\lfloor n_1\lfloor n_2-n_1},n_2\geq n_1\\ 0 \qquad\qquad\qquad, \text{otherwise}\end{cases}$$

SEMI RANDOM TELEGRAPH SIGNAL PROCESS

If N(t) represents the number of occurrence of a specified event in (0, t) and X(t) = $(-)^{N(t)}$, then {X(t)} is called a semi-random telegraph signal process.

1 RANDOM TELEGRAPH SIGNAL PROCESS

Definition

A random telegraph process is a discrete random process X(t) satisfying the following:

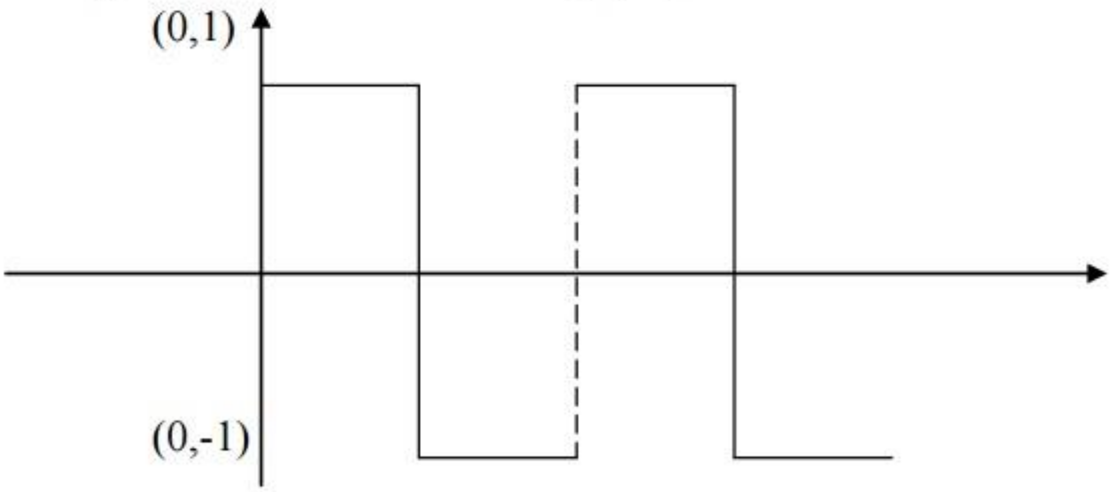
i. X(t) assumes only one of the two possible values 1 or –1 at any time 't'

ii. X(0) = 1 or –1 with equal probability 1/2

iii. The number of occurrence N(t) from one value to another occurring in any interval of length 't' is a Poisson process with rate λ, so that the probability of exactly 'r' transitions is

$$P[N(t)=r]=\frac{e^{-\lambda t}(\lambda t)^r}{r!}, r=0,1,2,...$$

A typical sample function of telegraph process.



Note: The process is an example for a discrete random process.

* Mean and Auto Correlation P{X(t) = 1} and P{X(t) = 1" for any t.

POISSON BINOMIAL PROCESS

If X(t) represents the no. of occurrences of certain even in (0, t), then the discrete random process {X(t)} is called the Poisson process, provided the following postulate are satisfied.

i. P[1 occurrence in $(t, t+\Delta t) = \lambda\Delta t$

ii. P[0 occurrence in $(t, t+\Delta t) = 1 - \lambda\Delta t$

iii. P[2 occurrence in $(t, t+\Delta t) = 0$

X(t) is independent of the no. of occurrences of the event in any interval prior and after the interval (0, t)

v. The Prob. That the events occurs a specified no. of times in (t0, t0+t) depends only on t, but not on t0.

Prob. Law for the Poisson Process {X(t)}

$$P\left[X(t)=n\right]=P_n(t)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$
$$n = 0, 1, 2, 3, \ldots$$

BINOMIAL PROCESS

Let Xn, n = 1, 2, 3, … be a Bernoulli Process and Sn denote the No. of the successes in the 1st n Bernoulli trails i.e., S

$$S_n = X_1 + X_2 + \ldots + X_n \; P\left[X_n = k\right] = \binom{n}{k} P^k q^{n-k}, \; k = 0, 1, 2, \ldots$$

Example:3.7.1

Suppose that customers arrive at a bank according to a Poisson Process with mean rate of 3 per minute. Find the Prob. That during a time interval of 2 minutes (i) exactly 4 customer arrive(ii)Greater than 4 customer arrive (iii) Fewer than 4 customer arrive.

$$\lambda = 3$$

$$P\left[X(t) = n\right] = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0,1,2\ldots$$

$$= \frac{E^{-3t}(3t)^n}{n!} \qquad n = 0,1,\ldots$$

P (Exactly 4 customers in 2 minutes)

$$= P\left[X(2) = 4\right] = \frac{e^{-664}}{4!} = 0.1338$$

(ii) P[more than 4 customers in 2 minutes]

$$= P\left[X(2) > 4\right] = 1 - P\left[X(2) \le 4\right]$$

$$= 1 - e^{-6}\left[1 + 6 + 62/_{2!} + 63/_{3!} + 64/_{4!}\right]$$

$$= 0.7149$$

(iii) P[Fewer than 4 customer arrive in 2 minutes]

$$= P\left[X(2) < 4\right]$$

$$= e^{-6}\left[1 + 6 + 62/_{2!} + 63/_{3!}\right]$$

$$= 0.1512$$

Example:3.7.2

If customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the Prob. that the interval 6/w two consecutive arrivals is (i) more than 1 minute (ii) B/W 1 & 2 minute (iii) 4 minutes or less

λ= 2

$$\lambda = 2$$

$$\text{(i) } P\left[T > 1\right] = \int_1^\infty 2e^{-2t}dt = 0.1353$$

$$\text{(ii) } P\left[1 < T < 2\right] = 0.1170$$

$$\text{(iii) } P\left[T \le 4\right] = \int_0^4 2e^{-2t}dt = 0.9996$$

TUTORIAL QUESTIONS

1.. The t.p.m of a Marko cain with three states 0,1,2 is P and the initial state distribution is Find (i)P[X2=3] ii)P[X3=2, X2=3, X1=3, X0=2]

2. Three boys A, B, C are throwing a ball each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. S.T.

the process is Markovian. Find the transition matrix and classify the states

3. A housewife buys 3 kinds of cereals A, B, C. She never buys the same cereal in successive weeks. If she buys cereal A, the next week she buys cereal B. However if she buys P or C the next week she is 3 times as likely to buy A as the other cereal. How often she buys each of the cereals?

4. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of week, the man tossed a fair die and drove to work if a 6 appeared. Find 1) the probability that he takes a train on the 3rd day. 2). The probability that he drives to work in the long run.

WORKED OUT EXAMPLES

Example:1.Let X_n denote the outcome of the n^{th} toss of a fair die.

Here $S = \{1, 2, 3, 4, 5, 6\}$

$T = \{1, 2, 3, \ldots\}$

$\therefore (X_n, n = 1, 2, 3, \ldots\}$ is a discrete random sequence.

Example:2 Given an example of stationary random process and justify your claim.

Solution:

Let us consider a random process X(t) = A as (wt + θ) where A &ω are custom uniformly distribution random Variable in the (0, 2π).

Since 'θ' is uniformly distributed in (0, 2π), we have

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, 0 < C < 2\pi \\ 0 \text{ ,otherwise} \end{cases}$$

$$\therefore E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$$

$$= \int_{0}^{2\pi} A\omega(\omega t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi}\left[\sin(\omega t + \theta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[\text{Sin}(2\pi + \omega t) - \text{Sin}(\omega t + 0)\right]$$

$$= \frac{A}{2\pi}[\text{Sin}\omega t - \sin\omega t]$$

$$= 0 \text{ constant}$$

Since E[X(t)] = a constant, the process X(t) is a stationary random process.

Example:3.which are not stationary .Examine whether the Poisson process {X(t)} gi

probability law P{X(t)=n] = $\frac{e^{-\lambda t}(\lambda t)}{\lfloor n}$, n = 0, 1, 2, ….

Solution

We know that the mean is given by

$$E\left[X(t)\right]=\sum_{n=0}^{\infty} nP_n(t)$$

$$=\sum_{n=0}^{\infty}\frac{ne^{-\lambda t}(\lambda t)^n}{\lfloor n}$$

$$=\sum_{n=1}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{\lfloor n-1}$$

$$=e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{\lfloor n-1}$$

$$=e^{-\lambda t}\left[\frac{\lambda t}{0!}+\frac{(\lambda t)^2}{1!}+\ldots\right]$$

$$=(\lambda t)e^{-\lambda t}\left(1+\frac{\lambda t}{\lfloor 1}+\frac{(\lambda t)^2}{\lfloor 2}+\ldots\right)$$

$$=(\lambda t)e^{-\lambda t}e^{\lambda t}$$

$$=\lambda t\text{, depends on t}$$

Hence Poisson process is not a stationary process.