Definition:

Let a function $f(t)$ be continuous and defined for +ve values of t. The Laplace transformation of $f(t)$ associates a function s defined by the equation.

$$\phi(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Here $\phi(s)$ is said to be the Laplace transform of $f(t)$ and It is denoted by $L[f(t)]$.

Thus $$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt, \quad t > 0$$

The Symbol L is called the Laplace transform operator.

Important formula:

1. $L[1] = \frac{1}{s}$ where $s > 0$

2. $L[t^n] = \frac{n!}{s^{n+1}}$ where $n = 0, 1, 2 \ldots$

3. $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ where n is not an integer

4. $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$

5. $L[e^{at}] = \frac{1}{s-a}$ where $s - a > 0$

9. $L[\cosh at] = \frac{s}{s^2-a^2}$ where $s^2 > a^2$

10. $L[a f(t) \pm b g(t)] = a L[f(t)] \pm b L[g(t)]$.

(Linearity property)

1. Prove that $L[e^{at}] = \frac{1}{s-a}$ Provided $s-a > 0$

Proof:

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at} dt \quad [\text{By definition}]$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{1}{s-a}$$

$$\therefore L[e^{at}] = \frac{1}{s-a}$$

2. Prove that $L[\cosh at] = \frac{s}{s^2-a^2}$

Solution:

$$L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{\quad}\right]$$

$$L[\cos at] = \int_0^{\infty} e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2} \right]_0^{\infty}$$

$$\left[\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \right]$$

$$= \frac{e^{-\infty}}{s^2+a^2} - \frac{e^{0}(-s)}{s^2+a^2}$$

$$= 0 + \frac{s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

$$L[\cos at] = \frac{s}{s^2+a^2}$$

Problem 1:

1. Find $L[5^t]$

Sol: We know that $L[e^{at}] = \frac{1}{s-a}$

$$L[5^t] = L[e^{\log 5^t}]$$

$$= L[e^{t \log 5}]$$

$L[\sin 5t \cdot \cos 2t] = \frac{1}{2} L[\sin 7t + \sin 3t]$

$$= \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right].$$

$$L[\sin 5t \cos 2t] = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

3. Find $L[\cosh 2t - \cos 2t]$.

<u>Sol</u>

$$L[\cosh 2t - \cos 2t] = \frac{s}{s^2-4} - \frac{s}{s^2+4}$$

4. Find $L[(\sin t - \cos t)^2]$.

$$L[(\sin t - \cos t)^2] = L[\sin^2 t + \cos^2 t - 2\sin t \cos t]$$

$$= L[1 - \sin 2t]$$

$$= L(1) - L(\sin 2t)$$

$$= \frac{1}{s} - \frac{2}{s^2+4}$$

5. Prove that $L(1) = \frac{1}{s}$.

<u>Sol</u>

$$L[1] = L[e^{0t}]$$

$$= \int_0^\infty e^{-st} e^{0t}\, dt.$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty} - e^{0}}{-s} = \frac{1}{s}.$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \int_0^\infty e^{-st} d[f(t)]$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty f(t)\, d(e^{-st})$$

$$\mathcal{L}\{f'(t)\} = 0 - e^0 f(0) - \int_0^\infty f(t)\, e^{-st}(-s)\, dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= s\mathcal{L}[f(t)] - f(0)$$

Similarly $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}[f(t) - s^2 f(0) - sf'(0) - f''(0)$$

and so on. In general.

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- s^{n-3} f''(0) - \dots - f^{n-1}(0)$$

Linearity Property:

If c_1 and c_2 are constants and $f_1(t)$ and $f_2(t)$ are given functions, then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

$$= c_1 \mathcal{L}[f_1(t)] + c_2 \mathcal{L}[f_2(t)]$$

1. __Example:__

Find $\mathcal{L}[e^{2t} + 3e^{-5t}]$

Sol
$$\mathcal{L}(e^{2t} + 3e^{-5t}) = \mathcal{L}(e^{2t}) + \mathcal{L}(3e^{-5t})$$
$$= \frac{1}{s-2} + 3\frac{1}{s+5}$$

2. Find $\mathcal{L}[\sinh 6t + 3e^{-5t} + \cos 5t)$

Sol $\mathcal{L}(\sinh 6t + 3e^{-5t} + \cos 5t]$
$$= \mathcal{L}(\sinh 6t) + \mathcal{L}(3e^{-5t}) + \mathcal{L}(\cos 5t)$$
$$= \frac{6}{s^2 - 6^2} + 3\frac{1}{s+5} + \frac{s}{s^2 + 5^2}$$
$$= \frac{6}{s^2 - 36} + \frac{3}{s+5} + \frac{s}{s^2 + 25}$$

3. Find $\mathcal{L}[\cos^2 3t]$.

Sol
$$\mathcal{L}(\cos^2 3t) = \mathcal{L}\left[\frac{1 + \cos 6t}{2}\right].$$
$$= \frac{1}{2}\left[\mathcal{L}(1) + \mathcal{L}(\cos 6t)\right].$$
$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 6^2}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{\ldots}\right]$$

Sol

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

Put $st = x$ when $t=0,\ x=0$

$\therefore s\,dt = dx$ when $t=\infty,\ x=\infty$

$$\therefore L[t^n] = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{1}{s}\, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx.$$

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \quad \left[\because \int_0^\infty x^n e^{-x} dx = \Gamma(n+1)\right]$$

when n is a +ve integer then $\Gamma(n+1) = n!$

$$\therefore L[t^n] = \frac{n!}{s^{n+1}}$$

1. Find $L[(t+2t^2)^2]$

Sol $L[(t+2t^2)^2] = L[t^2 + 4t^4 + 4t^3]$

$$= L[t^2] + 4L[t^4] + 4L[t^3]$$

$$= \frac{2}{s^3} + \frac{4(4!)}{s^5} + \frac{4(3!)}{s^4}$$

2. Find $L\left[\frac{1}{\sqrt{\pi t}}\right]$

Sol $L\left[\frac{1}{\sqrt{\pi t}}\right] = L\left[\frac{1}{\sqrt{\pi}} t^{-1/2}\right] = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\ }$

Proof: $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$\therefore L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\,dt.$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt, \quad s-a>0$$

$$= F(s-a)$$

$\therefore L[e^{at} f(t)] = F(s-a)$ where $F(s) = L[f(t)]$

The unit step function

The function is denoted by $H(t)$ and is defined as
$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

we also have $H(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$ where $a > 0$.

Second Shifting Theorem:

If $L[f(t)] = F(s)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L[G(t)] = e^{-as} F(s)$.

Proof:
$$L[G(t)] = \int_0^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

$$\therefore L[G(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)\,du$$

In $\int_0^\infty e^{-su} f(u)\,du$, u is dummy Variable.

Hence we can Replace it by the Variable t.

$$\therefore L[G(t)] = e^{-sa}\int_0^\infty e^{-st} f(t)\,dt$$

$$= e^{-sa} L[f(t)]$$

$$= e^{-sa} F(s).$$

1. Find $L[t^2 e^{-2t}]$

Sol $L[e^{-2t} t^2] = L(t^2)_{s \to s+2}$

$$= \left(\frac{2}{s^3}\right)_{s \to s+2}$$

$$= \frac{2}{(s+2)^3}$$

2. Find $L[e^{-t}(3\sinh 2t - 5\cosh 2t)]$

Sol $L[e^{-t}(3\sinh 2t - 5\cosh 2t)]$

$$= L[3\sinh 2t - 5\cosh 2t]_{s \to s+1}$$

$$= 3L(\sinh 2t)_{s \to s+1} - 5L(\cosh 2t)_{s \to s+1}$$

$$= \frac{6-5(s+1)}{(s+1)^2-4}$$

$$= \frac{-5s+1}{s^2+2s-3}.$$

3. Find $L[e^{-t}t^9]$

$$L[e^{-t}t^9] = L[t^9]_{s\to s+1}$$

$$= \left(\frac{9!}{s^{10}}\right)_{s\to s+1}$$

$$= \frac{9!}{(s+1)^{10}}$$

4. Find $L[\cosh t \sin 2t]$

Sol $L[\cosh t \cdot \sin 2t] = L\left[\left(\frac{e^t+e^{-t}}{2}\right)\sin 2t\right]$

$$= \frac{1}{2}L[e^t \sin 2t] + \frac{1}{2}L[e^{-t}\sin 2t] \longrightarrow (1)$$

Now $L[\sin 2t] = \frac{2}{s^2+4}$

$$\therefore L[e^t \sin 2t] = \frac{2}{(s-1)^2+4} \quad \text{—— (2)}$$

Also $L[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+4}$ —— (3)

Substituting (2) & (3) in (1) we get

$$0 \quad \text{if} \quad t < \frac{2\pi}{3}$$

Solution:

We know that by Second shifting property

if $L[f(t)] = F(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0 & t < a \end{cases}$

then $L[G(t)] = e^{-as} F(s)$ —— (1)

Here $F(t-a) = \cos\left(t - \frac{2\pi}{3}\right)$

(ie) $F(t) = \cos t$ and $a = \frac{2\pi}{3}$ —— (2)

$\therefore L[f(t)] = L[\cos t] = \frac{s}{s^2+1}$ —— (3)

Substituting (2) & (3) in (1) we get.

$$\therefore L[G(t)] = e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}$$

Prove that $L[H(t)] = \frac{2(1-e^{-\pi s})}{s^2+4}$ where

$$H(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & t < \pi \end{cases}$$

Sol

$$L[H(t)] = \int_0^\infty e^{-st} H(t)\, dt$$

$$= \int_0^\pi e^{-st} H(t)\, dt + \int_\pi^\infty e^{-st} H(t)\, dt$$

$$= \frac{2}{s^2+4} - \frac{2e^{-\pi s}}{s^2+4}$$

Change of Scale property:

If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt.$$

Put $at = x$ when $t = 0,\ x = 0$

$a\,dt = dx$ $\quad t = \infty,\ x = \infty$

$$\therefore L[f(at)] = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\, \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\, dx.$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\, dt$$

[$\because x$ is dummy Variable]

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\therefore L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

1. Find $L[f(t)]$ where $f(t) = \{\ 0$ when $0 < t < 2$

$$= \int_0^2 e^{-st} \cdot 0 \cdot dt + \int_2^\infty e^{-st} \cdot 2 \cdot dt$$

$$= 3 \left[\frac{e^{-st}}{-s} \right]_2^\infty$$

$$= 3 \left[\frac{e^{-\infty} - e^{-2s}}{-s} \right] = \frac{3e^{-2s}}{s}$$

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2. Find $L[f(t)]$, where $f(t) = \begin{cases} \cos t, & \text{when } 0 < t < \pi \\ \sin t, & \text{when } t > \pi \end{cases}$

Sol

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

$$= \left[\frac{e^{-st}}{s^2+1} (-s\cos t + \sin t) \right]_0^\pi$$

$$+ \left[\frac{e^{-st}}{s^2+1} (-s\sin t - \cos t) \right]_0^\pi$$

$$= \frac{e^{-s\pi} \cdot s}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-s\pi}}{s^2+1}$$

$$= \frac{1}{s^2+1} \left[s + e^{-\pi s} (s-1) \right]$$

Theorem:

If $L[f(t)] = F(s)$, then $L[t\, f(t)] = -\frac{d}{ds} F(s)$.

$$= \frac{d}{ds}\left\{ \int_0^\infty e^{-st} f(t)\,dt \right\}$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\left[t f(t)\right] dt$$

$$\frac{d}{ds}\left[F(s)\right] = -L\left[t f(t)\right]$$

$$\therefore \quad L\left[t f(t)\right] = -\frac{d}{ds} F(s) = -F'(s)$$

where $F(s) = L[f(t)]$.

1. Find $L[t \sin 2t]$

Sol We know that

$$L[t f(t)] = -\frac{d}{ds} F(s), \text{ where } F(s) = L[f(t)]$$

Here $f(t) = \sin 2t$

$$F(s) = L[f(t)] = L[\sin 2t] = \frac{2}{s^2+4}$$

(ii) $F(s) = \dfrac{2}{s^2+4}$

$$\therefore \quad L(t \sin 2t) = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\left[\frac{0 - 2(2s)}{}\right]$$

We know that $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$ where

$F(s) = L[f(t)]$. Here $f(t) = e^{-3t}$.

$$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}]$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s+3}\right)\right] = \frac{d}{ds}\left[\frac{0-1}{(s+3)^2}\right]$$

$$= \frac{2(s+3)}{(s+3)^4} = \frac{2}{(s+3)^3}$$

3. Find $L[t e^{-t} \cosh t]$

Sol $L[t e^{-t} \cosh t] = -\frac{d}{ds} L[e^{-t} \cosh t]$

$$= -\frac{d}{ds} L[\cosh t]_{s \to s+1}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2-1}\right)_{s \to s+1}$$

$$= -\left[\frac{(s^2-1) - s(2s)}{(s^2-1)^2}\right]_{s \to s+1}$$

$$= \left[\frac{-s^2+1+2s^2}{(s^2-1)^2}\right]_{s \to s+1}$$

$$= \left[s^2+1 \right.$$

4. Find $L[t^2\cos 3t]$.

Sol:

$$L[t^2\cos 3t] = (-1)^2 \frac{d^2}{ds^2} L[\cos 3t]$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2+9) - s(2s)}{(s^2+9)^2}\right]$$

$$L[t^2\cos 3t] = \frac{d}{ds}\left[\frac{9-s^2}{(s^2+9)^2}\right]$$

$$= \frac{(s^2+9)^2(-2s) - (9-s^2)\,2(s^2+9)\cdot 2s}{(s^2+9)^4}$$

$$= \frac{-2s(s^2+9) - 4s(9-s^2)}{(s^2+9)^3} = \frac{2s^3-54s}{(s^2+9)^3}$$

5. Find $L[(t\sin at)^2]$.

Sol

$$L[(t\sin at)^2] = L\left[t^2\left(\frac{1-\cos 2at}{2}\right)\right]$$

$$= \frac{1}{2}(-1)^2\frac{d^2}{ds^2}[L(1-\cos 2at)]$$

$$= \frac{1}{2}\frac{d^2}{ds^2}\left\{\frac{1}{s} - \frac{s}{s^2+4a^2}\right\}$$

$$= \frac{1}{2}\frac{d}{ds}\left\{\frac{-1}{s^2} + \frac{s^2-4a^2}{(\ldots)^2}\right\}$$

$$= \frac{1}{s^3} + \frac{s(12a^2 - s^2)}{(s^2+4a^2)^3}$$

Theorem:

If $L\{f(t)\} = F(s)$ and if $\frac{f(t)}{t}$ has a limit as $t \to 0$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$.

Proof

Given $F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$\therefore \int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds.$$

$$= \int_s^\infty ds \int_0^\infty e^{-st} f(t)\,dt$$

[$\because$ s & t are independent variables and hence the order of integration in the double integral can be interchanged).

$$\therefore \int_s^\infty F(s)\,ds = \int_0^\infty dt \int_s^\infty e^{-st} f(t)\,ds.$$

$$= \int_0^\infty f(t)\,dt \int_0^\infty e^{-st}\,ds$$

$$= \int_0^\infty f(t)\,dt \left[\frac{e^{-st}}{-t}\right]_s^\infty$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt$$

$= L\{f(t)\}$.

1. Find $L\left[\frac{1-e^{-t}}{t}\right]$

Sol $L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty L(1-e^{-t})\,ds$

$= \int_s^\infty \left[L(1) - L(e^{-t})\right] ds$

$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds$

$= \left[\log s - \log(s+1)\right]_s^\infty$

$= \left(\log \frac{s}{s+1}\right)_s^\infty$

$= \log\left(\frac{1}{1+\frac{1}{s}}\right)_s^\infty$

$= \log 1 - \log \frac{1}{1+\frac{1}{s}}$

$= 0 - \log\left(\frac{s}{s+1}\right)$

$= \log\left(\frac{s}{s+1}\right)^{-1} = \log\left(\frac{s+1}{s}\right)$

2. Using Laplace transform Prove that $\int_0^\infty \frac{1-\cos 2t}{t^2}\,dt = \pi$

Sol $\int_0^\infty \frac{1-\cos 2t}{t^2}\,dt = \left[L\left[\frac{1-\cos 2t}{t^2}\right]\right]_{s=0}$

$$= \int_S^\infty \left[\log s - \frac{1}{2}\log(s^2+4)\right]_S^\infty ds \Bigg]_{s=0}$$

$$= \left[\int_S^\infty \left[\log \frac{s}{\sqrt{s^2+4}}\right]_S^\infty ds\right]_{s=0}$$

$$= \left[\int_S^\infty \left(0 - \log \frac{s}{\sqrt{s^2+4}}\right) ds\right]_{s=0}$$

$$= \left[\int_S^\infty \left[\log \frac{\sqrt{s^2+4}}{s}\right] ds\right]_{s=0}$$

$$= \left[\frac{1}{2}\int_S^\infty \log\left(1+\frac{4}{s^2}\right) ds\right]_{s=0}$$

$$= \left\{\left[\frac{1}{2} s \log\left(1+\frac{4}{s^2}\right)\right]_S^\infty - \frac{1}{2}\int_S^\infty s \frac{1}{1+\frac{4}{s^2}}\left(\frac{-8}{s^3}\right) ds\right\}_{s=0}$$

$$= \left[0 - \frac{1}{2} s \log\left(\frac{s^2+4}{s^2}\right) + \frac{8}{2}\int_S^\infty \frac{1}{s^2+4} ds\right]_{s=0}$$

$$\int_0^\infty \frac{1-\cos 2t}{t^2} dt = \left[0 + 4\int_S^\infty \frac{1}{s^2+4} ds\right]_{s=0}$$

$= \pi$

3. Find $L\left[\frac{\cos 4t \sin 2t}{t}\right]$

Solution:

$$L\left[\frac{\cos 4t \sin 2t}{t}\right] = \int_s^\infty L(\cos 4t \sin 2t)\, ds.$$

$$= \int_s^\infty \left(\frac{\sin 6t - \sin 2t}{2}\right) ds.$$

$$L\left(\frac{\cos 4t \sin 2t}{t}\right) = \frac{1}{2}\int_s^\infty \left\{L[\sin 6t] - L[\sin 2t]\right\} ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{6}{s^2+36} - \frac{2}{s^2+4}\right] ds.$$

$$= \frac{1}{2}\left[6 \cdot \frac{1}{6}\tan^{-1}\left(\frac{s}{6}\right) - 2 \cdot \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\tan^{-1}(\infty) - \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{6}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{6}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right].$$

Theorem:

$$L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s} L[f(t)]$$

Proof: Let $\int^t f(x)\,dx = F(t)$

$$(ii)\quad L[f(t)] = s\,L\left[\int_0^t f(x)\,dx\right].$$

Example

1. Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$.

Sol:

$$L\left(\int_0^t t\cos t\,dt\right) = \frac{1}{s}L(t\cos t)$$

$$= \frac{1}{s}\left[-\frac{d}{ds}L(\cos t)\right]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$$

$$= \frac{1}{s}\left[-\frac{(s^2+1) - s\cdot 2s}{(s^2+1)^2}\right] = \frac{s^2-1}{s(s^2+1)^2}$$

$$\therefore L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

2. Find the Laplace transform of $\int_0^t te^{-t}\sin t\,dt$.

Sol

$$L(\sin t) = \frac{1}{s^2+1}$$

Hence $L\left[\int_0^t t e^{-t}\sin t\, dt\right] = \frac{1}{s} L\left(t e^{-t}\sin t\right)$

$$= \frac{1}{s} \cdot \frac{2(s+1)}{s^2+2s+2}$$

3. Find $L\left[\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right]$.

<u>Sol</u>

$$L\left[\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right] = \frac{1}{s} L\left[\frac{e^{-t}\sin t}{t}\right]$$

Now $\left[L\left[\frac{e^{-t}\sin t}{t}\right]\right] = \int_s^\infty L\left[e^{-t}\sin t\right] ds$

$$= \int_s^\infty \frac{ds}{(s+1)^2+1}$$

$$= \left[-\cot^{-1}(s+1)\right]_s^\infty$$

$$= 0 + \cot^{-1}(s+1)$$

$$\therefore L\left[\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right] = \frac{\cot^{-1}(s+1)}{s}$$

Initial Value Theorem:

If $L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

<u>Proof:</u> We know that $L[f'(t)] = sL[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$$= \lim_{S \to \infty} \int_0^{\infty} e^{-St} f'(t)\, dt$$

$$= 0 \quad \left[\because \lim_{S \to \infty} e^{-St} f'(t)\, dt = 0 \right]$$

$$\therefore \lim_{S \to \infty} S F(S) = f(0)$$

$$= \lim_{t \to 0} f(t)$$

---

1. Verify the initial value theorem for the function

$$f(t) = a e^{-bt}$$

Sol Initial value theorem is

$$\lim_{t \to 0} f(t) = \lim_{S \to \infty} S F(s)$$

Here $f(t) = a e^{-bt}$

$$\therefore F(s) = \mathcal{L}[f(t)]$$

$$= \mathcal{L}[a e^{-bt}]$$

$$= a \frac{1}{S+b}$$

$$\therefore \lim_{t \to 0} f(t) = \lim_{t \to 0} a e^{-bt} = a \quad \text{--- (A)}$$

$$\lim_{S \to \infty} S F(s) = \lim_{S \to \infty} S \cdot \frac{a}{S+b} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{S \to \infty} \frac{a}{1} \quad [\text{Using L' Hospital's rule}] \text{ --- (B)}$$

$$t \to \infty \qquad s \to 0$$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0)$

$$= sF(s) - f(0)$$

(ie) $sF(s) - f(0) = \int_0^\infty e^{-st} f'(t)\,dt$

(ie) $\lim_{s\to 0} \{sF(s) - f(0)\} = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt$

$$= \int_0^\infty f'(t)\,dt = [f(t)]_0^\infty$$

$$= \lim_{t\to\infty} f(t) - f(0)$$

(ie) $\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$.

Verify the initial and final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Solution:

$$L[f(t)] = L[1 + e^{-t}\sin t + e^{-t}\cos t]$$

$$F(s) = \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

Initial Value Theorem:

$$= \lim_{s\to\infty} \frac{2s^2+2s+4}{(s+1)^2+1} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{s\to\infty} \frac{4s+3}{2(s+1)} \left(\frac{\infty}{\infty}\right) = \frac{4}{2} = 2$$

L.H.S = R.H.S.

Hence initial value theorem is verified.

Final value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

L.H.S $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 1 + e^{-t}(\sin t + \cos t)$

$= 1 \; [\because e^{-\infty} = 0]$

R.H.S $\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$

$$= \lim_{s\to 0} 1 + \frac{s}{(s+1)^2+1} + \frac{s(s+1)}{(s+1)^2+1}$$

$$= 1 + \frac{0}{2} + \frac{0}{2} = 1$$

L.H.S = R.H.S.

Hence final value theorem is verified.

1. If $L[f(t)] = \frac{1}{s(s+1)(s+2)}$, find $\lim_{t\to 0} f(t)$ and $\lim_{t\to\infty} f(t)$

$t \to 0$ $\qquad$ $s \to \infty$

$= 0.$

By the final value theorem,

$$\lim_{t \to \infty} [f(t)] = \lim_{s \to 0} [sF(s)]$$

$$= \frac{1}{2}.$$

Inverse Laplace transforms.

We know that the Laplace transform of $f(t)$ is denoted by $L(f(t))$ and it is a function of s

(ie) $L(f(t)) = F(s)$

Conversely if $F(s)$ is the Laplace transform of $f(t)$ then $f(t)$ is the inverse Laplace transform of $F(s)$

(ie) $L^{-1}(F(s)) = f(t).$

For example:

$$L(e^{at}) = \frac{1}{s-a}$$

$$\Rightarrow L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

Now we tabulate the inverse Laplace transform of some important functions.

$F(s)$	$L^{-1}(F(s)) = f(t)$

(iv)	$\frac{1}{(s-a)^n}$	$e^{at}\frac{t^{n-1}}{(n-1)!}$
(v)	$\frac{a}{s^2+a^2}$	$\sin at$
(vi)	$\frac{s}{s^2+a^2}$	$\cos at$
(vii)	$\frac{a}{s^2-a^2}$	$\sinh at$
(viii)	$\frac{s}{s^2-a^2}$	$\cosh at$
(ix)	$\frac{1}{(s-a)^2+b^2}$	$\frac{1}{b}e^{at}\sin bt$
(x)	$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
(xi)	$\frac{s}{(s^2+a^2)^2}$	$\frac{t\sin at}{2a}$
(xii)	$\frac{1}{(s^2+a^2)^2}$	$\frac{\sin at - at\cos at}{2a^3}$

Properties of Inverse Laplace transform

1. $L^{-1}(F(s)+F_1(s)) = L^{-1}(F(s)) + L^{-1}(F_1(s))$

2. $L^{-1}(aF(s)) = a\,L^{-1}(F(s))$ 'a' is a constant.

3. $L^{-1}(aF(s)+bF_1(s)) = a\,L^{-1}(F(s)) + b\,L^{-1}(F_1(s))$

We know that

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$\therefore\; L^{-1}\left(\frac{1}{s-b}\right) = e^{bt}$$

(ii) Find $L^{-1}\left(\frac{1}{4s} + \frac{16}{1-s^2}\right)$

$$= L^{-1}\left(\frac{1}{4s}\right) + L^{-1}\left(\frac{16}{1-s^2}\right)$$

$$= \frac{1}{4} L^{-1}\left(\frac{1}{s}\right) + 16\, L^{-1}\left(\frac{1}{1-s^2}\right)$$

$$= \frac{1}{4} + 16\, L^{-1}\left(\frac{-1}{s^2-1}\right)$$

$$= \frac{1}{4} - 16\, L^{-1}\left(\frac{1}{s^2-1}\right)$$

$$= \frac{1}{4} - 16 \sinh t$$

2. Find $L^{-1}\left[\frac{s}{s^2+2} + \frac{1}{s^{5/2}}\right]$

$$\Rightarrow L^{-1}\left(\frac{s}{s^2+2}\right) + L^{-1}\left(\frac{1}{s^{5/2}}\right)$$

$$= L^{-1}\left(\frac{s}{s^2+(\sqrt{2})^2}\right) + \frac{t^{5/2-1}}{\left(\frac{5}{2}-1\right)!}$$

$$= \cos\sqrt{2}\,t + \frac{t^{3/2}}{\left(\frac{3}{2}\right)!} \qquad \left[\,(n+1)! = n\,\Gamma(n)\,\right]$$

$$\frac{3}{2}! = \left(\frac{1}{2}+1\right)! = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \cos\sqrt{2}\,t + t^{3/2}$$

$$\Rightarrow L^{-1}\left(\frac{2s}{4s^2+25}\right) - L^{-1}\left(\frac{5}{4s^2+25}\right) + L^{-1}\left(\frac{4s}{9-s^2}\right) - 18\,L^{-1}\left(\frac{1}{9-s^2}\right)$$

$$= L^{-1}\left(\frac{2s}{4\left(s^2+\frac{25}{4}\right)}\right) - L^{-1}\left(\frac{2\left(5/2\right)}{4\left(s^2+\frac{25}{4}\right)}\right) - 4L^{-1}\left(\frac{s}{s^2-9}\right)$$

$$+ 18\,L^{-1}\left(\frac{1}{s^2-9}\right)$$

$$= \frac{1}{2}L^{-1}\left(\frac{s}{s^2+\frac{25}{4}}\right) - \frac{1}{2}L^{-1}\left(\frac{\frac{5}{2}}{s^2+\frac{25}{4}}\right) - 4L^{-1}\left(\frac{s}{s^2-9}\right)$$

$$+ 18\,L^{-1}\left(\frac{3}{3(s^2-9)}\right)$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin\frac{5}{2}t - 4\cosh 3t + \frac{18}{3}\sinh 3t.$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin\frac{5}{2}t - 4\cosh 3t + 6\sinh 3t.$$

Inverse using first shifting theorem.

We know that

If $L(f(t)) = F(s)$ then $L(e^{-at}f(t)) = F(s+a)$

$$L^{-1}(F(s+a)) = e^{-at}f(t)$$

(ie) $L^{-1}(F(s+a)) = e^{-at}L^{-1}(F(s))$

---

1. Find $L^{-1}\left(\frac{s}{(s+3)^2+4}\right)$

$= e^{-3t} L^{-1}\left(\frac{s}{s^2+4}\right) - 3e^{-3t} L^{-1}\left(\frac{1}{s^2+4}\right)$

$= e^{-3t}\cos 2t - 3e^{-3t} \cdot \frac{1}{2}\sin 2t$

$= e^{-3t}\left(\cos 2t - \frac{3}{2}\sin 2t\right)$

2. Find $L^{-1}\left(\frac{s+1}{s^2+6s+25}\right)$

$L^{-1}\left(\frac{s+1}{s^2+6s+25}\right) = L^{-1}\left(\frac{s+1}{s^2+6s+9+16}\right)$

$= L^{-1}\left(\frac{s+1}{(s+3)^2+16}\right)$

$= L^{-1}\left(\frac{s+3-2}{(s+3)^2+16}\right)$

$= L^{-1}\left(\frac{s+3}{(s+3)^2+16}\right) - 2L^{-1}\left(\frac{1}{(s+3)^2+16}\right)$

$= e^{-3t} L^{-1}\left(\frac{s}{s^2+16}\right) - \frac{2e^{-3t}}{4} L^{-1}\left(\frac{4}{s^2+16}\right)$

$= e^{-3t}\cos 4t - \frac{2e^{-3t}}{4}\sin 4t$

$= e^{-3t}\left(\cos 4t - \frac{\sin 4t}{2}\right)$

$$= L^{-1}\left(\frac{s-2+s-9}{(s-2)^2+4}\right)$$

$$= L^{-1}\left(\frac{s-2}{(s-2)^2+4}\right) + L^{-1}\left(\frac{s-9}{(s-2)^2+4}\right)$$

$$= e^{2t} L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{s-2-7}{(s-2)^2+4}\right)$$

$$= e^{2t}\cos 2t + L^{-1}\left(\frac{s-2}{(s-2)^2+4}\right) - 7L^{-1}\left(\frac{1}{(s-2)^2+4}\right)$$

$$= e^{2t}\cos 2t + e^{2t} L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{7e^{2t}}{2} L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= e^{2t}\cos 2t + e^{2t}\cos 2t - \frac{7}{2} e^{2t}\sin 2t$$

$$= 2e^{2t}\cos 2t - \frac{7}{2} e^{2t}\sin 2t$$

Inverse using Second Shifting Theorem

$$L^{-1}\left(e^{-as}F(s)\right) = \begin{cases} f(t-a) & t > a \\ 0 & 0 \le t < a \end{cases}$$

(ie) $L^{-1}\left(e^{-as}F(s)\right) = L^{-1}\left(F(s)\right)_{t \to t-a}$

1. Find $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+13}\right]$

$$L^{-1}\left(\frac{e^{-2s}}{s^2 \ldots}\right) = L^{-1}\left(\frac{1}{\ldots}\right)$$

$$= \frac{e^{-2t}}{3} L^{-1}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{e^{-2t}}{3} \sin 3t$$

$$\therefore L^{-1}\left(\frac{e^{-2s}}{s^2+4s+13}\right) = \left(\frac{e^{-2t}}{3} \sin 3t\right)_{t \to t-2.}$$

$$\therefore L^{-1}\left(\frac{e^{-2s}}{s^2+4s+13}\right) = \frac{e^{-2(t-2)}}{3} \sin 3(t-2).$$

Inverse Laplace using derivatives:

$$L^{-1}(f'(s)) = -t\, L^{-1}(f(s))$$

$$\Rightarrow L^{-1}(f(s)) = \frac{-1}{t} L^{-1}(f'(s))$$

1. Find $L^{-1}\left(\log\left(1+\frac{w^2}{s^2}\right)\right)$

$$f(s) = \log\left(1+\frac{w^2}{s^2}\right) = \log\left(\frac{s^2+w^2}{s^2}\right)$$

$$= \log(s^2+w^2) - \log s^2$$

$$f'(s) = \frac{2s}{s^2+w^2} - \frac{2s}{s^2}$$

$$f'(s) = \frac{2s}{s^2+w^2} - \frac{2}{s}$$

$$L^{-1}(f'(s)) = L^{-1}\left(\frac{2s}{\quad} - \frac{2}{\quad}\right)$$

$$= 2\cos\omega t - 2.$$

$$L^{-1}(F(s)) = -\frac{L^{-1}(F'(s))}{t}$$

$$L^{-1}\left(\log\left(1+\frac{\omega^2}{s^2}\right)\right) = -\left(\frac{2\cos\omega t - 2}{t}\right)$$

2. Find the inverse Laplace of

$$\tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)$$

Let $F(s) = \tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)$

$$F'(s) = \frac{1}{1+\left(\frac{a}{s}\right)^2}\left(-\frac{a}{s^2}\right) - \frac{1}{1+\left(\frac{s}{b}\right)^2}\left(\frac{1}{b}\right)$$

$$F'(s) = \frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}$$

$$L^{-1}(F'(s)) = L^{-1}\left(\frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}\right)$$

$$= -\sin at - \sin bt$$

$$L^{-1}(F(s)) = -\frac{L^{-1}(F'(s))}{t}.$$

$$L^{-1}\left(\tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)\right) = \frac{-\sin at - \sin bt}{t}$$

$$= -\left(\frac{\sin at + \sin bt}{t}\right)$$

Laplace transform of some functions which contains Square term in their denominator.

Example:

Find $L^{-1}\left(\dfrac{s+6}{(s^2+12s+40)^2}\right)$

$$f'(s) = \frac{s+6}{(s^2+12s+40)^2}$$

$$F(s) = \int \frac{s+6}{(s^2+12s+40)^2}\, ds.$$

Let $z = s^2+12s+40$

$dz = (2s+12)\, ds$

$\dfrac{dz}{2} = (s+6)\, ds.$

$$F(s) = \int \frac{\frac{dz}{2}}{z^2}$$

$$= \frac{1}{2}\int \frac{dz}{z^2} = -\frac{1}{2z} = \frac{-1}{2(s^2+12s+40)}$$

$$\therefore L^{-1}(F(s)) = \frac{-1}{2} L^{-1}\left(\frac{1}{s^2+12s+40}\right)$$

$$= \frac{-1}{2} L^{-1}\left(\frac{1}{s^2+12s+36+4}\right)$$

$$= \frac{-1}{} L^{-1}\left(\frac{1}{}\right)$$

$$= -\frac{1}{4} e^{-6t} \sin 2t.$$

$$L^{-1}(F(s)) = -\frac{1}{4} e^{-6t} \sin 2t.$$

$$L^{-1}(F'(s)) = -t\, L^{-1}(F(s))$$

$$\therefore L^{-1}\left(\frac{s+6}{(s^2+12s+40)^2}\right) = -\frac{t}{4} e^{-6t} \sin 2t.$$

Find $L^{-1}\left(\frac{s}{(s^2+w^2)^2}\right)$

$$F'(s) = \frac{s}{(s^2+w^2)^2}$$

$$F(s) = \int \frac{s\,ds}{(s^2+w^2)^2}.$$

$$z = s^2+w^2$$

$$dz = 2s\,ds$$

$$\frac{dz}{2} = s\,ds$$

$$F(s) = \int \frac{\frac{dz}{2}}{z^2} = \frac{-1}{2z} = -\frac{1}{2(s^2+w^2)}$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{-1}{2(s^2+w^2)}\right)$$

$$= \frac{-1}{2} L^{-1}\left(\frac{1}{s^2+w^2}\right) = \frac{-1}{2w} \sinh wt.$$

$$L^{-1}(sF(s)) = \frac{d}{dt} L^{-1}(F(s))$$

In general $L^{-1}(s^n F(s)) = \frac{d^n}{dt^n} L^{-1}(F(s))$

Ex:

1. Find $L^{-1}\left(\frac{s}{(s+2)^2+4}\right)$

$$L^{-1}\left(s.\frac{1}{(s+2)^2+4}\right) = \frac{d}{dt} L^{-1}\left(\frac{1}{(s+2)^2+4}\right)$$

$$= \frac{d}{dt}\left(e^{-2t} L^{-1}\left(\frac{1}{s^2+4}\right)\right)$$

$$= \frac{d}{dt}\left(e^{-2t}\frac{\sin 2t}{2}\right)$$

$$= \frac{1}{2}\left(2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t\right)$$

$$= e^{-2t}\cos 2t - e^{-2t}\sin 2t.$$

2. Find $L^{-1}\left(\frac{s}{(s+3)^{5/2}}\right)$

$$L^{-1}\left(\frac{s}{(s+3)^{5/2}}\right) = \frac{d}{dt} L^{-1}\left(\frac{1}{(s+3)^{5/2}}\right)$$

$$= \frac{d}{dt} e^{-3t} L^{-1}\left(\frac{1}{s^{5/2}}\right)$$

$$= \frac{d}{dt} e^{-3t} \frac{t^{3/2}}{}$$

$$= \frac{2}{\sqrt{\pi}} \frac{d}{dt}\left(e^{-3t} t^{3/2}\right)$$

$$= \frac{2}{\sqrt{\pi}} \left[e^{-3t} \frac{3}{2} t^{1/2} - 3e^{-3t} t^{3/2} \right]$$

$$= \frac{2}{\sqrt{\pi}} e^{-3t} \sqrt{t} \left[\frac{3}{2} - 3t \right]$$

$$= 2e^{-3t} \sqrt{\frac{t}{\pi}} \left[\frac{3-6t}{2} \right]$$

$$= 3e^{-3t} \sqrt{\frac{t}{\pi}} \left[1 - 2t \right]$$

Inverse Laplace for the functions of the form $\frac{F(s)}{s}$

$$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t L^{-1}(F(s))\,dt$$

1. Find $L^{-1}\left(\frac{1}{s(s+1)^3}\right)$

$$L^{-1}\left(\frac{1}{s} \cdot \frac{1}{(s+1)^3}\right) = \int_0^t L^{-1}\left(\frac{1}{(s+1)^3}\right) dt$$

$$= \int_0^t e^{-t} L^{-1}\left(\frac{1}{s^3}\right) dt$$

$$= \int_0^t e^{-t} \frac{t^2}{2}\, dt$$

$$= \left[-\frac{e^{-t}}{2} \left(t^2 + 2t + 2 \right) \right]_0^t$$

$$= -\frac{e^{-t}}{2}\left(t^2 + 2t + 2 \right) + \frac{e^0 (2)}{2}$$

$$= -\frac{e^{-t}}{2}\left(t^2 + 2t + 2 \right) + 1$$

2. Find $L^{-1}\left(\frac{1}{s^2 (s+a)} \right)$

$$= L^{-1}\left(\frac{1}{s}\left(\frac{1}{s} \frac{1}{(s+a)} \right) \right)$$

$$= \int_0^t L^{-1}\left(\frac{1}{s}\left(\frac{1}{s+a} \right) \right) dt$$

$$= \int_0^t \left(\int_0^t L^{-1}\left(\frac{1}{s+a} \right) dt \right) dt = \int_0^t \left(\int_0^t e^{-at} dt \right) dt$$

$$= \int_0^t \left. \frac{e^{-at}}{-a} \right]_0^t dt = -\frac{1}{a} \int_0^t \left(e^{-at} - 1 \right) dt$$

$$= -\frac{1}{a} \left[\frac{e^{-at}}{-a} - t \right]_0^t$$

$$= -\frac{1}{a} \left[\frac{e^{-at}}{-a} - t + \frac{1}{a} \right]$$

$$= \frac{e^{-at}}{a^2} + \frac{t}{a} - \frac{1}{a^2}$$

$$= \frac{1}{2a}\int_0^t L^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right)dt$$

Consider $L^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right)$

$$F'(s) = \frac{2as}{(s^2+a^2)^2}$$

$$\Rightarrow F(s) = \int \frac{2as}{(s^2+a^2)^2}\,ds$$

$z = s^2+a^2$

$dz = 2s\,ds$

$$F(s) = a\int \frac{dz}{z^2} = \frac{-a}{z} = \frac{-a}{s^2+a^2}$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{-a}{s^2+a^2}\right) = -\sin at$$

$$L^{-1}(F'(s)) = -t\,L^{-1}(F(s))$$

$$L^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right) = -t(-\sin at)$$

$$\therefore L^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right) = t\sin at$$

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a}\int_0^t t\sin at\,dt$$

$= \frac{1}{...}\left[-t\cos at + \sin at\right]^t$

Proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$F(as) = \int_0^\infty e^{-ast} f(t)\, dt$$

$z = at$ when $t = 0$, $z = 0$

$\frac{dz}{a} = dt$ $\quad t = \infty$, $z = \infty$

$$\therefore F(as) = \int_0^\infty e^{-sz} f\left(\frac{z}{a}\right) \frac{dz}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-sz} f\left(\frac{z}{a}\right) dz$$

$$= \frac{1}{a} L\left(f\left(\frac{z}{a}\right)\right)$$ (z is a dummy Variable)

$$F(as) = \frac{1}{a} L\left(f\left(\frac{t}{a}\right)\right)$$

$$\Rightarrow L^{-1}(F(as)) = \frac{1}{a} f(t/a)$$

$$\Rightarrow L^{-1}(F(as)) = \frac{1}{a} L^{-1}(F(s))_{t \to t/a}$$

1. Find $L^{-1}\left[\frac{s}{s^2a^2+b^2}\right]$

$$\Rightarrow L^{-1}\left[\frac{as}{a(s^2a^2+b^2)}\right] = \frac{1}{a} L^{-1}\left[\frac{as}{(as)^2+b^2}\right]$$

$$F(as) = \frac{as}{(as)^2+b^2}, \quad F(s) = \frac{s}{s^2+b^2}$$

$$\therefore L^{-1}\left(\frac{s}{s^2a^2+b^2}\right) = \frac{1}{a}L^{-1}\left(\frac{as}{(as)^2+b^2}\right)$$

$$= \frac{1}{a^2}\cos\frac{bt}{a}$$

2. Find $L^{-1}\left(\frac{s}{2s^2-8}\right)$

$$L^{-1}\left(\frac{s}{2s^2-8}\right) = L^{-1}\left(\frac{2s}{4s^2-16}\right)$$

$$= L^{-1}\left(\frac{2s}{(2s)^2-16}\right)$$

$F(as) = \frac{2s}{(2s)^2-16}$ Here $a=2$.

$$F(s) = \frac{s}{s^2-16}$$

$$L^{-1}(F(2s)) = L^{-1}\left(\frac{2s}{(2s)^2-16}\right)$$

$$= \frac{1}{2}L^{-1}(F(s))_{t\to t/2}$$

$$= \frac{1}{2}L^{-1}\left(\frac{s}{s^2-16}\right)_{t\to t/2}$$

$$= \left(\frac{\cosh 4t}{2}\right)_{t\to t/2}$$

$$= \frac{1}{2}\cosh\left(\frac{t}{2}\right)$$

1. Find the inverse Laplace of $\dfrac{s^2+s-2}{s(s+3)(s-2)}$

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$\Rightarrow s^2+s-2 = A(s+3)(s-2) + B(s-2)s + Cs(s+3)$$

Put $s=0$	Put $s=2$	Put $s=-3$
$-2=-6A$	$4=10C$	$4=15B$
$\Rightarrow A = \frac{1}{3}$	$C = \frac{4}{10}$	$B = \frac{4}{15}$
	$= \frac{2}{5}$	

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{1}{3s} + \frac{4}{15}\frac{1}{s+3} + \frac{2}{5}\left(\frac{1}{s-2}\right)$$

$$\therefore L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right) = \frac{1}{3}L^{-1}\left(\frac{1}{s}\right) + \frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right) + \frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right)$$

$$= \frac{1}{3} + \frac{4e^{-3t}}{15} + \frac{2e^{2t}}{15}$$

2. Find the inverse Laplace of $\dfrac{1+2s}{(s+2)^2(s-1)^2}$.

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$\Rightarrow 1+2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2$$

$\Rightarrow A - 2C = 0$ $\qquad\qquad \Rightarrow 2A - C = 0$

$A = 0 \quad C = 0$

$$\mathcal{L}^{-1}\left(\frac{1+2s}{(s+2)^2(s-1)^2}\right) = \mathcal{L}^{-1}\left(\frac{-\frac{1}{3}}{(s+2)^2} + \frac{\frac{1}{3}}{(s-1)^2}\right)$$

$$= -\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$= -\frac{1}{3}te^{-2t} + \frac{1}{3}te^{t}$$

$2A - \frac{1}{3} - 4C = \frac{4}{3}$

3. Find $\mathcal{L}^{-1}\left(\frac{s}{(s^2+a^2)(s^2+b^2)}\right)$

$$\frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

$$s = (As+B)(s^2+b^2) + (Cs+D)(s^2+a^2)$$

$$s = (A+C)s^3 + (B+D)s^2 + (Ab^2+Ca^2)s + Bb^2 + Da^2$$

$$\left.\begin{aligned} A+C&=0 \\ Ab^2+Ca^2&=1 \end{aligned}\right\} - (1) \qquad \left.\begin{aligned} Bb^2+Da^2&=0 \\ B+D&=0 \end{aligned}\right\} - (2)$$

(By equating the co-efficients)

Using the equation (1) & Solving for A & C

we get $A = \frac{1}{a^2-b^2}$ $\quad C = \frac{-1}{a^2-b^2}$

$$-\frac{1}{a^2-b^2}L^{-1}\left(\frac{s}{s^2+b^2}\right)$$

$$=\frac{1}{a^2-b^2}\cos at-\frac{1}{a^2-b^2}\cos bt$$

$$=\frac{1}{a^2-b^2}(\cos at-\cos bt).$$

Convolution

The Convolution of two function $f(t)$ & $g(t)$ for $t \geq 0$ is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$$

Convolution theorem:

If $f(t)$ & $g(t)$ are functions defined for $t \geq 0$ then $L(f(t) * g(t)) = L(f(t)) \cdot L(g(t))$

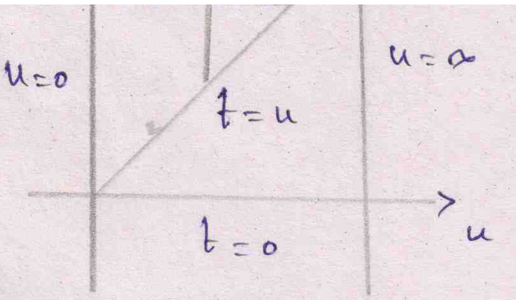
$$L(f(t) * g(t)) = \int_0^\infty e^{-st}(f(t) * g(t))\, dt.$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)\, g(t-u)\, du\right] dt.$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt.$$

Changing the order of integration we get.

$$= \int_0^\infty f(u) \int_0^\infty e^{-(u+v)s} g(v)\, dv\, du$$



$$= \int_0^\infty f(u) e^{-us} \int_0^\infty e^{-vs} g(v)\, dv\, du$$

$$= \int_0^\infty f(u) e^{-us}\, du \int_0^\infty e^{-vs} g(v)\, dv$$

$$= \int_0^\infty e^{-st} f(t)\, dt \int_0^\infty e^{-st} f(t)\, dt$$

[u & v are dummy variable]

$$\Rightarrow L(f(t) * g(t)) = L(f(t)) \cdot L(g(t)) = F(s) \cdot G(s)$$

$$\therefore L(f(t) * g(t)) = F(s) \cdot G(s)$$

1. Using Convolution theorem find $L^{-1}\left(\dfrac{1}{(s+a)(s+b)}\right)$

We know that $L^{-1}(F(s) \cdot G(s)) = L^{-1}(F(s)) * L^{-1}(G(s))$

$$F(s) = \frac{1}{s+a} \qquad G(s) = \frac{1}{s+b}$$

$$\therefore L^{-1}\left(\frac{1}{s+a} \cdot \frac{1}{s+b}\right) = L^{-1}\left(\frac{1}{s+a}\right) * L^{-1}\left(\frac{1}{s+b}\right)$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)}\, du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{1}{-(a-b)}\right]$$

$$= \frac{e^{-bt}}{a-b}\left[1 - e^{-at}e^{bt}\right]$$

$$= \frac{1}{a-b}\left[e^{-bt} - e^{-at}\right]$$

2. Find $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right) = L^{-1}\left(\frac{s}{s^2+a^2}\right) * L^{-1}\left(\frac{s}{s^2+b^2}\right)$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \cos(au+bt-bu) + \cos(au-bt+bu)\, du$$

$$\left[2\cos A\cos B = \cos(A+B) - \cos(A-B)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(bt+au-bu)}{a-b} + \frac{\sin(au+bu-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

3. Use Convolution to show that

$$\int_0^t \sin u \cos(t-u)\,du = \frac{t\sin t}{2}$$

$$(f(t) * g(t)) = \int_0^t f(u)\,g(t-u)\,du$$

$$\therefore \int_0^t \sin u \cos(t-u)\,du = \sin t * \cos t$$

$$\sin t * \cos t = \int_0^t \sin u \cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \sin(u+t-u) + \sin(u-t+u)\,du$$

$$= \frac{1}{2}\int_0^t (\sin t + \sin(2u-t))\,du$$

$$= \frac{1}{2}\left[u\sin t - \frac{\cos(2u-t)}{2}\right]_0^t$$

$$= \frac{1}{2}\left[t\sin t - \frac{\cos t}{2} + \frac{\cos t}{2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

3. Use Convolution to show that

$$\int_0^t \sin u \cos(t-u)\,du = \frac{t\sin t}{2}.$$

$$(f(t) * g(t)) = \int_0^t f(u)\,g(t-u)\,du$$

$$\therefore \int_0^t \sin u \cos(t-u)\,du = \sin t * \cos t$$

$$\sin t * \cos t = \int_0^t \sin u \cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \sin(u+t-u) + \sin(u-t+u)\,du$$

$$= \frac{1}{2}\int_0^t (\sin t + \sin(2u-t))\,du$$

$$= \frac{1}{2}\left[u\sin t - \frac{\cos(2u-t)}{2}\right]_0^t$$

$$= \frac{1}{2}\left[t\sin t - \frac{\cos t}{2} + \frac{\cos t}{2}\right]$$

in solving the differential equation where ordinary methods fail.

Consider the linear differential equation with constant which is of the form

$$\frac{d^n y}{dx^n} + C_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + C_{n-1}\frac{dy}{dt} + C_n y = F(t)$$

where $F(t)$ is a function of independent variable t

$C_1, C_2 \dots C_n$ are constants

1. Solve $(D^2+4D+4)y = 4e^{-2t}$, $y(0) = -1$ & $y'(0) = 4$

The given equation can be written as

$$D^2y + 4Dy + 4y = 4e^{-2t}$$

$$\Rightarrow y'' + 4y' + 4y = 4e^{-2t}$$

Taking Laplace transform on both sides

we get $L(y'') + 4L(y') + 4L(y) = 4L(e^{-2t})$

$$\Rightarrow s^2 L(y) - sy(0) - y'(0) + 4sL(y) - 4y(0) + 4L(y) = 4L(e^{-2t})$$

$$\Rightarrow s^2\bar{y} + s - 4 + 4s\bar{y} + 4 + 4\bar{y} = \frac{4}{s+2} \quad \text{where } \bar{y} = L(y)$$

$$\Rightarrow (s^2+4s+4)\bar{y} = \frac{4}{s+2} - s$$

$$\Rightarrow (s+2)^2\bar{y} = \frac{4}{s+2} - s$$

$$= 4e^{-2t} L^{-1}\left(\frac{1}{s^3}\right) - L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$$

$$y(t) = 4e^{-2t}\frac{t^2}{2!} - L^{-1}\left(\frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}\right)$$

$$= 2e^{-2t}t^2 - L^{-1}\left(\frac{1}{s+2}\right) + L^{-1}\left(\frac{2}{(s+2)^2}\right)$$

$$= 2t^2e^{-2t} - e^{-2t} + 2e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$y(t) = 2t^2e^{-2t} - e^{-2t} + 2e^{-2t}t$$

2. Solve $(D^2+9)y = \cos 2t$ if $y(0)=1$, $y\left(\frac{\pi}{2}\right) = -1$

The equation can be written as

$$y''+9y=\cos 2t$$

Taking Laplace transform on both sides we get

$$L(y'') + 9L(y) = L(\cos 2t)$$

$$s^2L(y) - sy(0) - y'(0) + 9L(y) = \frac{s}{s^2+4}$$

Since $y'(0)$ is not given take $y'(0)=c$, c is constant.

$$s^2\bar{y} - s - c + 9\bar{y} = \frac{s}{s^2+4}$$

~~Sin~~ $(s^2+9)\bar{y} - s - c = \frac{s}{s^2+4}$

$(s^2+9)\bar{y}$

$$= \frac{1}{5} L^{-1}\left(\frac{5s}{(s^2+9)(s^2+4)}\right) + \cos 3t + \frac{c \sin 3t}{3}$$

$$= \frac{1}{5} L^{-1}\left(\frac{s^3+9s-s^3-4s}{(s^2+9)(s^2+4)}\right) + \cos 3t + \frac{c}{3}\sin 3t$$

$$= \frac{1}{5} L^{-1}\left(\frac{s(s^2+9)-s(s^2+4)}{(s^2+9)(s^2+4)}\right) + \cos 3t + \frac{c}{3}\sin 3t$$

$$y = \frac{1}{5}\left[L^{-1}\left(\frac{(s(s^2+9))}{(s^2+9)(s^2+4)}\right) - L^{-1}\left(\frac{s(s^2+4)}{(s^2+4)(s^2+9)}\right)\right] + \cos 3t + \frac{c \sin 3t}{3}$$

$$= \frac{1}{5}\left[L^{-1}\left(\frac{s}{s^2+4}\right) - L^{-1}\left(\frac{s}{s^2+9}\right)\right] + \cos 3t + \frac{c \sin 3t}{3}$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{c \sin 3t}{3}$$

$$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c \sin 3t}{3}$$

To find c

given $y\left(\frac{\pi}{2}\right) = -1$

$$\therefore\ y\left(\frac{\pi}{2}\right) = \frac{1}{5}\cos 2\left(\frac{\pi}{2}\right) + \frac{4}{5}\cos 3\left(\frac{\pi}{2}\right) + \frac{c \sin \frac{3\pi}{2}}{3}$$

$$= -1$$

$$\Rightarrow\ -\frac{1}{5} - \frac{c}{3} = -1$$

$$\frac{}{5} + \frac{}{5} \quad \frac{}{5} \quad \frac{}{3}$$

Solve $(D^2+n^2)x = a\sin(nt+\alpha)$

$x = Dx = 0$ at $t=0$

$x'' + n^2 x = a\sin(nt+\alpha)$

Taking Laplace transform

$$\left[s^2\bar{x} - s\,x(0) - x'(0)\right] + n^2\bar{x} = a\left[L(\sin nt\cos\alpha) + L(\cos nt\sin\alpha)\right]$$

$$(s^2+n^2)\,\bar{x} = a\cos\alpha\,\frac{n}{s^2+n^2} + a\sin\alpha\,\frac{s}{s^2+n^2}$$

$$\bar{x} = an\cos\alpha\,\frac{1}{(s^2+n^2)^2} + a\sin\alpha\,\frac{s}{(s^2+n^2)^2}$$

$$x = an\cos\alpha\,L^{-1}\left(\frac{1}{(s^2+n^2)^2}\right) + a\sin\alpha\,L^{-1}\left(\frac{s}{(s^2+n^2)^2}\right)$$

$$= an\cos\alpha\,\frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha\,\frac{t}{2n}\sin nt$$

$$= \frac{a\{\sin nt\cos\alpha - nt\cos(nt+\alpha)\}}{2n^2}$$

Simultaneous Linear equations:

We can use Laplace transform to solve the Simultaneous linear equations with Constant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

1. Solve $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$

$$\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0 \qquad x(0) = 0, \quad y(0) = 0$$

The equation can be written as

$$3x' + y' + 2x = 1$$

$$x' + 4y' + 3y = 0$$

Taking Laplace transform

$$3L(x') + L(y') + 2L(x) = L(1)$$

$$L(x') + 4L(y') + 3L(y) = 0$$

$$\Rightarrow 3(sL(x) - x(0)) + sL(y) - y(0) + 2L(x) = \frac{1}{s}$$

$$sL(x) - x(0) + 4sL(y) - 4y(0) + 3L(y) = 0$$

$$\Rightarrow 3s\bar{x} + s\bar{y} + 2\bar{x} = \frac{1}{s}$$

$$s\bar{x} + 4s\bar{y} + 3\bar{y} = 0$$

$$\Rightarrow (3s+2)\bar{x} + s\bar{y} = \frac{1}{s}$$

$$s\bar{x} + (4s+3)\bar{y} = 0$$

Solving for $\bar{x}$ & $\bar{y}$

$\bar{x} = \begin{vmatrix} \frac{1}{s} & s \\ \end{vmatrix}$ $4s+3$

$$\bar{x} = \frac{4s+3}{s(11s^2+17s+6)}$$

$$\Rightarrow \quad x = L^{-1}\left(\frac{4s+3}{s(11s^2+17s+6)}\right)$$

$$\frac{4s+3}{s(11s^2+17s+6)} = \frac{4s+3}{s(s+1)(11s+6)}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{11s+6}$$

$$\therefore \quad 4s+3 = A(s+1)(11s+6) + Bs(11s+6) + Cs(s+1)$$

Put $s=0$

$$3 = 6A$$

$$A = 1/2$$

Put $s=-1$

$$-1 = 5B$$

$$\Rightarrow B = -1/5$$

Put $s=1$

$$7 = 34A + 17B + 2C$$

$$2C = -10 + \frac{17}{5}$$

$$-2C = -\frac{33}{5}$$

$$C = -\frac{33}{10}$$

$$x = L^{-1}\left(\frac{4s+3}{s(11s^2+17s+6)}\right)$$

$$= \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{3}{10}e^{-\frac{6t}{11}}$$

$$\bar{Y} = \frac{\begin{vmatrix} 3s+2 & 1/s \\ s & 0 \end{vmatrix}}{\begin{vmatrix} 3s+2 & s \\ s & 4s+3 \end{vmatrix}}$$

$$= \frac{-1}{(3s+2)(4s+3) - s^2}$$

$$\bar{Y} = \frac{-1}{11s^2 + 17s + 6}$$

$$Y = -L^{-1}\left(\frac{1}{11s^2+17s+6}\right)$$

$$= -L^{-1}\left(\frac{1}{(11s+6)(s+1)}\right)$$

Consider $\frac{1}{(11s+6)(s+1)} = \frac{A}{11s+6} + \frac{B}{s+1}$

$\Rightarrow 1 = A(s+1) + B(11s+6)$

Put $s = -1$

$\Rightarrow 1 = -5B$

$B = -1/5$

Put $s = 0$

$1 = A + 6B$

$$= L^{-1}\left(\frac{11}{5(11)}\frac{1}{s+\frac{6}{11}}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{5}L^{-1}\left(\frac{1}{s+\frac{6}{11}}\right) - \frac{1}{5}e^{-t}$$

$$= \frac{1}{5}e^{-\frac{6t}{11}} - \frac{1}{5}e^{-t}$$

$$\therefore\ y = -\frac{1}{5}e^{-\frac{6t}{11}} + \frac{1}{5}e^{-t}.$$

2. Solve $\frac{dx}{dt} + y = \sin t$; $\frac{dy}{dt} + x = \cos t$ with $x = 2$ and $y = 0$ when $t = 0$.

Sol The given differential equation Can be written as

$$x'(t) + y(t) = \sin t$$

$$y'(t) + x(t) = \cos t$$

Taking Laplace transform on both sides we get

$$L[x'(t)] + L[y(t)] = L[\sin t]$$

$$L[y'(t)] + L[x(t)] = L[\cos t]$$

$$sL[x(t)] - x(0) + L[y(t)] = \frac{1}{s^2+1}$$

$$sL[y(t)] - y(0) + L[x(t)] = \frac{s}{s^2+1}$$

$$s\bar{x} + \bar{y} = \frac{1}{s^2+1} + 2 \quad \text{———} (1)$$

$$\bar{x} + s\bar{y} = \frac{s}{s^2+1} \quad \text{———} (2)$$

Solving (1) & (2) Simultaneously we get

$$\bar{x} = \frac{\begin{vmatrix} \frac{1}{s^2+1}+2 & 1 \\ \frac{s}{s^2+1} & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ 1 & s \end{vmatrix}}$$

$$= \frac{s\left(2 + \frac{1}{s^2+1}\right) - \frac{s}{s^2+1}}{s^2-1}$$

$$= \frac{s\left[2(s^2+1)+1\right] - s}{(s^2+1)(s^2-1)}$$

$$= \frac{s\ (2s^2+2)}{(s^2+1)\ (s+1)(s-1)} = \frac{2s}{s^2-1}$$

(ie) $L[x(t)] = \dfrac{2s}{s^2-1}$

$$x(t) = 2 L^{-1}\left(\frac{s}{s^2-1}\right)$$

$$x(t) = 2\cosh t$$

$= \ |\ s \ \ \frac{1}{\ } \ + 2 \ |$

$$= \frac{s^2+3}{(s^2+1)(1-s^2)}$$

(ii) $L[y(t)] = \dfrac{s^2+3}{(s^2+1)(1-s^2)}$

$$y(t) = L^{-1}\left[\frac{s^2+3}{(s^2+1)(1-s^2)}\right] = L^{-1}\left[\frac{s^2+3}{(s^2+1)(1+s)(1-s)}\right]$$

$$= \frac{A}{1+s} + \frac{B}{1-s} + \frac{Cs+D}{s^2+1}$$

$$s^2+3 = A(1-s)(s^2+1) + B(1+s)(s^2+1) + (Cs+D)(1+s)(1-s)$$

Putting $s=1$, $4B=4 \Rightarrow B=1$

Putting $s=-1$, $4A=4 \Rightarrow A=1$

Putting $s=0$, $A+B+D=3 \Rightarrow D=1$

Equating Coefficient of s^3, $-A+B-C=0 \Rightarrow C=0$

$$\therefore \frac{s^2+3}{(s^2+1)(1+s)(1-s)} = \frac{1}{s+1} + \frac{1}{1-s} + \frac{1}{s^2+1}$$

$$L^{-1}\left[\frac{s^2+3}{(s^2+1)(1+s)(1-s)}\right] = L^{-1}\left[\frac{1}{s+1} + \frac{1}{1-s} + \frac{1}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

1. Find the inverse Laplace transform of $\frac{1}{(s+1)(s+2)}$

Sol Given $L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

$$1 = A(s+2) + B(s+1)$$

put $s=-1$ $\Rightarrow$ $1 = A$

put $s=-2$ $\Rightarrow$ $B = -1$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{-1}{s+2}\right]$$

$$= e^{-t} - e^{-2t}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$$

2. Find the Laplace transform of $\frac{t}{e^t}$

Sol Given $L\left[\frac{t}{e^t}\right] = L[te^{-t}]$

$$L[te^{-t}] = -\frac{d}{ds} L[e^{-t}]$$

$$= -\frac{d}{ds}\left[\frac{1}{s+1}\right]$$

Sol $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = ae^{0} = a \longrightarrow (1)$$

$$F(s) = L[f(t)] = L[ae^{-bt}] = a\frac{1}{s+b}$$

$$sF(s) = \frac{sa}{s+b} = \frac{\cancel{s}a}{\cancel{s}(1+b/s)}$$

$$\lim_{s\to\infty} sF(s) = \frac{a}{1} = a \longrightarrow (2)$$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$\therefore$ Initial value theorem is verified.

4. Find the Laplace transform of $f(t) = \frac{1-e^{-t}}{t}$

Sol $L[f(t)] = L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty L[1-e^{-t}]\,ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= \left[\log s - \log(s-1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty$$

$$= 0 - \log\left(\frac{s}{s-1}\right)$$

$$= \left(\log\left(\frac{s}{s-1}\right)\right)^{-1}$$

$$L\left[\frac{1-e^{-t}}{t}\right] = \log\frac{s-1}{s}$$

5. Find the Laplace transform of $f(t) = t\cosh t$

Sol

$$L[f(t)] = L\left[t \cdot \frac{e^{t}+e^{-t}}{2}\right]$$

$$= \frac{1}{2}\left\{L[te^{t}] + L[te^{-t}]\right\}$$

$$= \frac{1}{2}\left[-\frac{d}{ds}L[e^{t}] + \left(-\frac{d}{ds}L[e^{-t}]\right)\right]$$

$$= \frac{1}{2}\left[-\frac{d}{ds}\left(\frac{1}{s-1}\right) - \frac{d}{ds}\left[\frac{1}{s+1}\right]\right]$$

$$= \frac{1}{2}\left[-\left(\frac{-1}{(s-1)^2}\right] - \left(\frac{-1}{(s+1)^2}\right)\right]$$

$$L[t\cosh t] = \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right]$$

6. Find the inverse Laplace transform of $\frac{e^{-\pi s}}{(s-1)^2}$.

Sol

$$L^{-1}[f(t)] = L^{-1}\left[\frac{e^{-\pi s}}{(s-1)^2}\right]$$

$$= e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

Sol ~~w.k.T~~ $\cancel{L\{t\,f(t)\} = -\frac{d}{ds} f(s)}$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L[\sin t]\, ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$L\left[\frac{\sin t}{t}\right] = \cot^{-1}\left(\frac{s}{1}\right)$$

8. Evaluate $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$

Sol

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right] = L^{-1}\left[\frac{1}{(s^2+6s+9)+4}\right]$$

$$= L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$

$$= e^{-3t} L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= e^{-3t}\cdot\frac{\sin 2t}{2}.$$

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L\{1-\cos t\}\, ds$$

$$= \int_s^\infty \left(L\{1\} - L\{\cos t\}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log s - \log(s^2+1)^{1/2}\right]_s^\infty$$

$$= \left[\log\left[\frac{s}{\sqrt{s^2+1}}\right]\right]_s^\infty$$

$$= \log\left[\frac{1}{\sqrt{1+1/s^2}}\right]_s^\infty$$

$$= \log 1 - \log\frac{1}{\sqrt{1+1/s^2}}$$

$$= 0 - \log\frac{s}{\sqrt{s^2+1}} = \log\left(\frac{s}{\sqrt{(s^2+1)}}\right)^{-1}$$

$$= \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

10. Find the Laplace transform of the function

$$f(t) = \begin{cases} 1, & t=0 \\ 0, & t\neq 0 \end{cases}$$

Sol

Here $f(t)=1$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s}\left[0-1\right]$$

$$= \frac{1}{s}; \quad s>0$$

$$L[1] = \frac{1}{s}$$